ln L is the logarithm of the likelihood function

We wrote the log-likelihood function for the general linear model

$$Y = X\beta + e, \quad e \sim \underline{N}(0, \sigma^2 I_T)$$

$$\ln L = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)$$

<u>example</u>

1. $$y_t = \beta + e_t \qquad e_t \sim N(0, \sigma^2)$$

$$\ln L = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum(y_t - \beta)^2$$

$$\theta = \begin{bmatrix} \beta \\ \sigma^2 \end{bmatrix}$$

$$\frac{d\ln L}{d\beta} = \frac{1}{\sigma^2}\sum_{i=1}^{T}(y_t - \beta)$$

$$\frac{d\ln L}{d\sigma^2} = -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{T}(y_t - \beta)^2$$

$$I(\theta) \equiv I\begin{pmatrix} \beta \\ \sigma^2 \end{pmatrix}$$

$$2 \times 2$$

- $$\frac{\partial^2 \ln L}{\partial \beta^2} = \frac{\partial}{\partial \beta}\left[\frac{1}{\sigma^2}\sum y_t - \frac{1}{\sigma^2} T \cdot \beta\right]$$
$$= -\frac{T}{\sigma^2}$$

$$E - \frac{\partial^2 \ln L}{\partial \beta^2} = \frac{T}{\sigma^2}$$

- $$\frac{\partial^2 \ln L}{\partial (\sigma^2)^2} = \frac{T}{2\sigma^4} - \frac{1}{\sigma^6}\sum (y_t - \beta)^2$$

because $\frac{\partial}{\partial x}\left(\frac{1}{2x^2} = \frac{1}{2}x^{-2}\right) =$
$-2 \cdot \frac{1}{2} \cdot x^{-3}$

$$E - \frac{\partial^2 \ln L}{\partial (\sigma^2)^2} = -\left[\frac{T}{2\sigma^4} - \frac{1}{\sigma^6} \cdot T\sigma^2\right]$$

because $E \sum (y_t - \beta)^2 = E \sum e_t^2$
$= T\sigma^2$

$$= -\left[\frac{T}{2\sigma^4} - \frac{T}{\sigma^4}\right] = \frac{T}{2\sigma^4}$$

$$I\begin{pmatrix}\beta \\ \sigma^2\end{pmatrix} = \begin{bmatrix} \frac{T}{\sigma^2} & 0 \\ 0 & \frac{T}{2\sigma^4} \end{bmatrix}$$

$$I^{-1}\begin{pmatrix}\beta \\ \sigma^2\end{pmatrix} = \begin{bmatrix} \sigma^2/T & 0 \\ 0 & \frac{2\sigma^4}{T} \end{bmatrix}$$

$\frac{\sigma^2}{T}$ is the minimum variance an estimator of β can have

$\frac{2\sigma^4}{T}$ is the minimum variance, an estimator of σ^2 can have.

- If we estimate β by OLS or ML we would get

$b = \tilde{\beta}_{ML} = \frac{\sum y_t}{T}$, $var(b) = \frac{1}{T^2} \cdot \sum var\, y_t = \frac{1}{T^2} T\sigma^2$

$= \frac{\sigma^2}{T}$ ✓

MLE examples.

mple of size T

$$\begin{matrix} \text{vote for A} \\ \text{vote for B} \end{matrix} \Rightarrow Y_i = \begin{cases} 1 & \text{vote for A} \\ 0 & \text{vote for B} \end{cases}$$

θ: probability individual i votes $Y_i = 1$

$$P(Y_i = 1) = \theta$$

$$P(Y_i = 0) = 1 - \theta$$

$y_i \sim B(1, \theta)$

$$f(y_i) = \begin{cases} \theta & \text{if } y_i = 1 \\ 1-\theta & \text{if } y_i = 0 \end{cases}$$

$$\underbrace{f(y_i) = \theta^{y_i}(1-\theta)^{1-y_i}}$$

$$\underline{l(y_1 \ldots y_T)} = \prod_{i=1}^{T} f(y_i) = f(y_1) \cdot f(y_2) \cdots f(y_T) =$$

$$= \prod_{i=1}^{T} \theta^{y_i}(1-\theta)^{1-y_i}$$

$$= \theta^{\Sigma y_i}(1-\theta)^{T - \Sigma y_i}$$

$$\log l = \Sigma y_i \log \theta + (T - \Sigma y_i) \log(1-\theta)$$

$$\frac{\partial \log l}{\partial \theta} = 0 \iff \frac{\Sigma y_i}{\theta} - \frac{T - \Sigma y_i}{1-\theta} = 0$$

$$\frac{\Sigma y_i}{\theta} = \frac{T - \Sigma y_i}{1-\theta}$$

$$\hat{\theta}_{MLE} = \frac{\Sigma y_i}{T} = \bar{y}$$

$$E\,\hat{\theta}_{MLE} = \frac{\cancel{T}\cdot\theta}{\cancel{T}} = \theta$$

$$\text{var}\,\hat{\theta} = \frac{1}{T\cancel{^2}}\cancel{T}\cdot\theta(1-\theta) = \boxed{\frac{\theta(1-\theta)}{T}}$$

$$\frac{\partial^2 \log l}{\partial \theta^2} = -\frac{\Sigma y_i}{\theta^2} - \frac{T - \Sigma y_i}{(1-\theta)^2}$$

$$I = E\left[-\frac{\partial^2 \log l}{\partial \theta^2}\right] = E\left[\frac{\Sigma y_i}{\theta^2} + \frac{T - \Sigma y_i}{(1-\theta)^2}\right]$$

$$= \frac{T\cancel{\theta}}{\theta^{\cancel{2}}} + \frac{T\cancel{(1-\theta)}}{(1-\theta)^{\cancel{2}}}$$

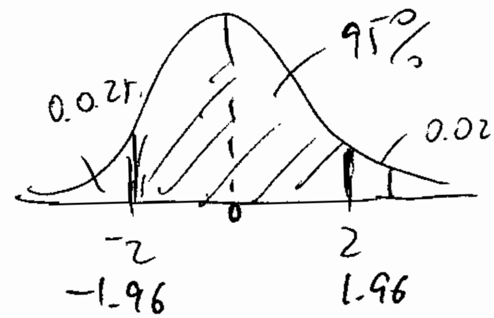
$$= \frac{T}{\theta} + \frac{T}{1-\theta} = \frac{T}{\theta(1-\theta)}$$

$$\text{var}(\hat{\theta}) = \underline{I^{-1}} = \boxed{\frac{\theta(1-\theta)}{T}}.$$

$$\sqrt{T}\,(\hat\theta-\theta) \underset{a}{\sim} N\left(0,\ \theta(1-\theta)\right)$$

$$\hat\theta \overset{a}{\sim} N\left(\theta,\ \frac{\theta(1-\theta)}{T}\right)$$

$$\frac{\hat\theta-\theta}{\sqrt{\frac{\theta(1-\theta)}{T}}} \overset{a}{\sim} N(0,1)$$



$$|\hat\theta-\theta| < 2\sqrt{\frac{\theta(1-\theta)}{T}}$$

$$\theta \in \left[\hat\theta \pm 2\sqrt{\frac{\theta(1-\theta)}{T}}\right]$$ with proba 95%

but θ is not known, replace by $\hat\theta$

$$\left[\theta \in \left(\hat\theta \pm 2\sqrt{\frac{\hat\theta(1-\hat\theta)}{T}}\right)\right]$$

where $\hat\theta = \bar y$

$$\max_{0<\theta<1} \theta(1-\theta) = \frac{1}{4}$$

$$\bar y \pm \frac{1}{\sqrt{T}}$$

Ex 2. $Y_1, \dots Y_T$ iid $\sim P(\lambda)$

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$$f(y_i;\lambda) = \exp(-\lambda) \frac{\lambda^{y_i}}{y_i!}$$

$$l(y;\lambda) = \prod_{i=1}^{T} f(y_i;\lambda)$$

$$= \prod_{i=1}^{T} \left[\exp(-\lambda) \frac{\lambda^{y_i}}{y_i!}\right] = \exp(-T\lambda) \frac{\lambda^{\Sigma y_i}}{\prod_{i=1}^{T} y_i!}$$

$$\log l(y;\lambda) = -T\lambda + \Sigma y_i \log\lambda - \log\left(\prod_{i=1}^{T} y_i!\right)$$

$$\frac{\partial \log l}{\partial \lambda} = 0 \iff -T + \frac{\Sigma y_i}{\hat\lambda} = 0$$

$$\hat\lambda = \frac{\Sigma y_i}{T} = \bar y$$

$$E\hat\lambda = E\bar y = \lambda$$

mean $= \lambda$
variance $= \lambda$

$$\text{var}(\hat\lambda) = \text{var}(\bar y) = \frac{1}{T}\text{var}(y_i) = \frac{\lambda}{T}$$

$$I: \quad \frac{\partial^2 \log l(y_i;\lambda)}{\partial\lambda^2} = -\frac{\Sigma y_i}{\lambda^2}$$

$$I(\lambda) = E\left(-\frac{\partial^2 \log l(y;\lambda)}{\partial\lambda^2}\right) = E\left[\frac{\Sigma y_i}{\lambda^2}\right] = \frac{T\lambda}{\lambda^2} = \frac{T}{\lambda}$$

var $\hat{\lambda} = I(\lambda)^{-1} = \frac{\lambda}{T}$

$$\sqrt{T}\,(\hat{\lambda} - \lambda) \overset{a}{\sim} N(0, \lambda)$$

$$\hat{\lambda} \underset{a}{\sim} N\left(\lambda, \frac{\lambda}{T}\right)$$

$$\left[\hat{\lambda} \pm 2\sqrt{\frac{\hat{\lambda}}{T}}\right]$$

$$\left[\hat{\lambda} \pm \frac{2}{\sqrt{T}}\sqrt{\hat{\lambda}}\right]$$

Fisher Theorem $\hat{m}$ and s^2 are independent.

$$\frac{(n-1)\,s^2}{\sigma^2} \sim \chi^2(n-1) \quad \text{Cochrane.}$$

$$\log l = -\frac{T}{2}\log\sigma^2 - \frac{T}{2}\log 2\pi - \frac{1}{2\sigma^2}\Sigma(y_t-\mu)^2$$ 6

$$\frac{\partial \log l}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{T}(y_i-\mu)$$

$$\frac{\partial \log l}{\partial \sigma^2} = -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{T}(y_i-\mu)^2$$

to see that $I\begin{pmatrix}\mu\\ \sigma^2\end{pmatrix} = \begin{bmatrix} \frac{T}{\sigma^2} & 0 \\ 0 & \frac{T}{2\sigma^4}\end{bmatrix}$

$$E\left[-\frac{\partial^2 \log l(\mu,\sigma^2)}{\partial \mu^2}\right] = \frac{T}{\sigma^2}$$

$$E\,-\frac{\partial^2 \log l(\mu,\sigma^2)}{(\partial\sigma^2)^2} = \frac{T}{2\sigma^4}$$

$$\hat{\sigma} = \frac{\Sigma(y_i-\hat{\mu})^2}{T}$$

$$I^{-1}(\mu,\sigma^2) = \begin{bmatrix} \frac{\sigma^2}{T} & 0 \\ 0 & \frac{2\sigma^4}{T}\end{bmatrix}$$

$$\operatorname{var}\begin{pmatrix}\hat{\mu}\\ s^2\end{pmatrix} = \begin{bmatrix} \frac{\sigma^2}{T} & 0 \\ 0 & \frac{2\sigma^4}{T-1}\end{bmatrix}$$

$$s^2 = \frac{\Sigma(y_i-\hat{\mu})^2}{T-1}$$

$$\operatorname{var} s^2 = \operatorname{var} \frac{1}{T-1}\sigma^2\chi^2_{(T-1)} = \frac{1}{(T-1)^2}\sigma^4\, 2(T-1)$$