Consider the simple regression model: $y_t = \beta_0 + \beta_1 x_t + u_t$, $t=1,\dots,T$ 1.

Using

$$X = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_T \end{bmatrix}, \qquad y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}$$

we have

$$(X'X) = \begin{bmatrix} T & \Sigma x_t \\ \Sigma x_t & \Sigma x_t^2 \end{bmatrix}$$

and

$$(X'X)^{-1} = \frac{1}{T\Sigma x_t^2 - (\Sigma x_t)^2} \begin{bmatrix} \Sigma x_t^2 & -\Sigma x_t \\ -\Sigma x_t & T \end{bmatrix}$$

$$X'y = \begin{bmatrix} \Sigma y_t \\ \Sigma x_t y_t \end{bmatrix}$$

Thus,

$$\hat{\beta} = (X'X)^{-1}X'y$$

$$= \frac{1}{T\Sigma x_t^2 - (\Sigma x_t)^2} \begin{bmatrix} \Sigma x_t^2 & -\Sigma x_t \\ -\Sigma x_t & T \end{bmatrix} \begin{bmatrix} \Sigma y_t \\ \Sigma x_t y_t \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{\Sigma x_t^2 \Sigma y_t - \Sigma x_t \Sigma x_t y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2} \\[2ex] \dfrac{T\Sigma x_t y_t - \Sigma x_t \Sigma y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

We need to prove that

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = \frac{\Sigma x_t^2 \Sigma y_t - \Sigma x_t \Sigma x_t y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2}$$

and

$$\hat{\beta}_1 = \frac{\Sigma(x_t - \bar{x})(y_t - \bar{y})}{\Sigma(x_t - \bar{x})^2} = \frac{T\Sigma x_t y_t - \Sigma x_t \Sigma y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2}$$

Now

$$\hat{\beta}_1 = \frac{\Sigma(x_t - \bar{x})(y_t - \bar{y})}{\Sigma(x_t - \bar{x})^2} = \frac{\Sigma x_t y_t - \bar{y}\Sigma x_t - \bar{x}\Sigma y_t + T\bar{x}\bar{y}}{\Sigma x_t^2 - 2\bar{x}\Sigma x_t + T\bar{x}^2}$$

$$= \frac{\Sigma x_t y_t - \frac{\Sigma y_t}{T}\Sigma x_t - \frac{\Sigma x_t}{T}\Sigma y_t + \frac{T\,\Sigma x_t \Sigma y_t}{T^2}}{\Sigma x_t^2 - 2\frac{\Sigma x_t}{T}\Sigma x_t + T\left(\frac{\Sigma x_t}{T}\right)^2}$$

$$= \frac{\Sigma x_t y_t - \Sigma x_t \Sigma y_t / T}{\Sigma x_t^2 - (\Sigma x_t)^2 / T}$$

$$= \frac{T\,\Sigma x_t y_t - \Sigma x_t \Sigma y_t}{T\,\Sigma x_t^2 - (\Sigma x_t^2)}$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1\bar{x} = \bar{y} - \left[\frac{T\Sigma x_t y_t - \Sigma x_t \Sigma y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2}\right]\bar{x}$$

$$= \frac{\bar{y}[T\Sigma x_t^2 - (\Sigma x_t)^2] - \bar{x}[T\Sigma x_t y_t - \Sigma x_t \Sigma y_t]}{T\Sigma x_t^2 - (\Sigma x_t)^2}$$

$$= \frac{\Sigma y_t \Sigma x_t^2 - \frac{\Sigma y_t (\Sigma x_t)^2}{T} - \Sigma x_t \Sigma x_t y_t + \frac{(\Sigma x_t)^2 \Sigma y_t}{T}}{T\Sigma x_t^2 - (\Sigma x_t)^2}$$

$$= \frac{\Sigma x_t^2 \Sigma y_t - \Sigma x_t \Sigma x_t y_t}{T\Sigma x_t^2 - (\Sigma x_t)^2}$$