SIMULTANEOUS EQUATIONS

we have system of equations: (c_t, y_t demeaned ⇒ no intercept)

(1) $$c_t = \gamma y_t + e_t$$

(2) $$y_t = c_t + i_t$$ ← Identity

with $e_t \sim N(0, \sigma^2)$

EVEN IF WE ASSUME c_t and i_t fixed or stochastic but not correlated with e_t

Y_t IS STOCHASTIC, CORRELATED WITH e_t

$$y_t = \gamma y_t + e_t + i_t$$

$$(1-\gamma) y_t = i_t + e_t$$

$$y_t = (1-\gamma)^{-1} i_t + (1-\gamma)^{-1} e_t$$

We will show that OLS applied to (1) is NOT CONSISTENT

$$\hat{\gamma}_{OLS} = \frac{\sum c_t y_t}{\sum y_t^2} = \frac{\sum (\gamma y_t + e_t) y_t}{\sum y_t^2}$$

$$\hat{c} = \gamma + \frac{\sum e_t y_t}{\sum y_t^2}$$

to check for bias we would have to take expectations, but on the r.h.s. we have ratio of 2 r.v. We evaluate plims AND OBTAIN A RESULT ON CONSISTANCY (⇒ ASYMPTOTIC)

$$= \gamma + \frac{\text{plim}\left(\frac{\sum e_t y_t}{T}\right)}{\text{plim}\left(\frac{\sum y_t^2}{T}\right)}$$

$$\text{plim}\left(\frac{\sum e_t y_t}{T}\right) = \text{plim}\left(\frac{\sum e_t\left((1-\gamma)^{-1} i_t + (1-\gamma)^{-1} e_t\right)}{T}\right)$$

$$= (1-\gamma)^{-1}\left[\text{plim}\left(\frac{\sum e_t i_t}{T}\right) + \text{plim}\left(\frac{\sum e_t^2}{T}\right)\right]$$

- as i_t uncorrelated with e_t $\text{plim}\left(\frac{\sum e_t i_t}{T}\right) = 0$
- $\text{plim}\left(\frac{\sum e_t^2}{T}\right) = \sigma_e^2$

- $\text{plim}\left(\frac{\sum y_t^2}{T}\right) = \text{var}(y_t) = m_{YY}$

Finally:

$$\text{plim}\ \hat{\gamma} = \gamma + \frac{(1-\gamma)^{-1}\sigma_e^2}{m_{YY}} \neq \gamma$$

$\Rightarrow$ **OLS estimator of γ in (1) IS NOT CONSISTENT**

NOTE: WE HAVE 2 DEPENDENT VARIABLES C_t, y_t and one explanatory i_t. We can eventually add an intercept β to (1)

WE WILL BUILD A REDUCED FORM:

ALL DEPENDENT ON THE LHS EXPRESSED IN TERMS OF ALL EXPLANATORY ON THE R.H.S

I.E:

first equation:

$$c_t = \beta + \gamma\,(c_t + i_t) + e_t$$

$$c_t = \frac{\beta}{1-\gamma} + \frac{\gamma}{1-\gamma}\, i_t + \frac{e_t}{1-\gamma}$$

$$* \quad c_t = \Pi_{10} + \Pi_{11}\, i_t + v_t$$

$$\Pi_{10} = \frac{\beta}{1-\gamma} \qquad \Pi_{11} = \frac{\gamma}{1-\gamma} \qquad v_t = \frac{1}{1-\gamma}\, e_t \sim N\left(0, \frac{\sigma^2}{(1-\gamma)^2}\right)$$

second equation

$$y_t = \beta + \gamma y_t + e_t + i_t$$

$$y_t = \frac{\beta}{1-\gamma} + \frac{1}{1-\gamma}\, i_t + \frac{e_t}{1-\gamma}$$

$$* \quad y_t = \Pi_{20} + \Pi_{21}\, i_t + v_t$$

$$\Pi_{20} = \frac{\beta}{1-\gamma} \qquad \Pi_{21} = \frac{1}{1-\gamma}, \qquad v_t = \frac{1}{1-\gamma}\, e_t \sim N\left(0, \frac{\sigma^2}{(1-\gamma)^2}\right)$$

Note that: conditional,

$$v_t \sim N\left(0, \frac{\sigma^2}{(1-\gamma)^2}\right) \Rightarrow c_t \sim N\left[\left(\frac{\beta}{1-\gamma} + \frac{\gamma}{1-\gamma} i_t\right), \frac{\sigma^2}{(1-\gamma)^2}\right]$$

$$v_t \sim N\left(0, \frac{\sigma^2}{(1-\gamma)^2}\right) \Rightarrow y_t \sim N\left[\left(\frac{\beta}{1-\gamma} + \frac{1}{1-\gamma} i_t\right), \frac{\sigma^2}{(1-\gamma)^2}\right]$$

We will estimate:

(3) $\quad c_t = \Pi_{10} + \Pi_{11} i_t + v_t$

(4) $\quad y_t = \Pi_{20} + \Pi_{21} i_t + v_t$

BY INDIRECT LEAST SQUARES

1) OLS applied to (3) $\Rightarrow \hat{\Pi}_{10}, \hat{\Pi}_{11}$

2) OLS to (4) $\Rightarrow \hat{\Pi}_{20}, \hat{\Pi}_{21}$

and since

$$\frac{\Pi_{11}}{\Pi_{21}} = \frac{\gamma/1-\gamma}{1/1-\gamma} = \gamma \Rightarrow \frac{\hat{\Pi}_{11}}{\hat{\Pi}_{21}} \Rightarrow \tilde{\gamma}_{ILS}$$

$$\frac{\Pi_{20}}{\Pi_{21}} = \frac{\beta/1-\gamma}{1/1-\gamma} = \beta \Rightarrow \frac{\hat{\Pi}_{20}}{\hat{\Pi}_{21}} \Rightarrow \tilde{\beta}_{ILS}$$

CONSISTENT, BIASED IN SMALL SAMPLES

Simultaneous equation approach to demand-supply.

demand: $p_t = \beta_{11} + \gamma_{12} q_t^d + \beta_{12} ps_t + \beta_{13} di_t$

Supply: $q_t^s = \beta_{21} + \gamma_{21} p_t + \beta_{24} pf_t$

→ these 2 equations represent an economic model

at equilibrium: $q_t^s = q_t^d = q_t$ so that:

$$p_t = \beta_{11} + \gamma_{12} q_t + \beta_{12} ps_t + \beta_{13} di_t$$

$$q_t = \beta_{21} + \gamma_{21} p_t \qquad + \beta_{24} pf_t$$

note: p_t and q_t are determined WITHIN the system

ps_t, di_t, pf_t OUTSIDE

THE ECONOMETRIC MODEL:

$$p_t = \beta_{11} + \gamma_{12} q_t + \beta_{12} ps_t + \beta_{13} di_t + e_{1t}$$

$$q_t = \beta_{21} + \gamma_{21} p_t \qquad + \beta_{24} pf_t + e_{2t}$$

where errors are contemporaneously correlated across equations (recall SUR)

Assumption:

$$E(e_{it}) = 0 \quad \text{for } i=1,2$$

$$var(e_{it}) = \sigma_{ii} \text{ or } \sigma_i^2 \quad \text{for } i=1,2$$

$$cov(e_{it}\, e_{jt}) = \sigma_{ij} \neq 0 \quad \text{for } i=1,2$$

for any two errors observed at the same time t, we have

$$cov\begin{pmatrix} e_{1t} \\ e_{2t} \end{pmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \quad \left(\text{or: } \begin{matrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{matrix}\right)$$

We assume:

$$\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}\right]$$

- FIRST STEP:

REDUCED FORM

$$P_t = \beta_{11} + \gamma_{12}\left(\beta_{21} + \gamma_{21} P_t + \beta_{24} P_{f_t} + e_{2t}\right) + \beta_{12} P_{s_t} +$$

$$+ \beta_{13} d_{i_t} + e_{1t}$$

solve for p_t:

$$p_t = \frac{\beta_{11} + \gamma_{12}\beta_{21}}{1-\gamma_{12}\gamma_{21}} + \frac{\beta_{12}}{1-\gamma_{12}\gamma_{21}} ps_t + \frac{\beta_{13}}{1-\gamma_{12}\gamma_{21}} di_t + $$

$$+ \frac{\gamma_{12}\beta_{24}}{1-\gamma_{12}\gamma_{21}} pf_t + \frac{\gamma_{12}e_{2t} + e_{1t}}{1-\gamma_{12}\gamma_{21}}$$

- $p_t = \Pi_{11} + \Pi_{12} ps_t + \Pi_{13} di_t + \Pi_{14} pf_t + v_{1t}$,

$$v_{1t} \sim N(0, \sigma_{v_1}^2)$$

substitute expression on the top to:

$$q_t = \beta_{21} + \gamma_{21} p_t + \beta_{24} pf_t + e_{2t}$$

solve for q_t

- $q_t = \Pi_{21} + \Pi_{22} ps_t + \Pi_{23} di_t + \Pi_{24} pf_t + v_{2t}$

$$v_{2t} \sim N(0, \sigma_{v_2}^2)$$

where

$$\Pi_{12} = \frac{\beta_{12}}{1-\gamma_{12}\gamma_{21}}, \quad \Pi_{22} = \frac{\gamma_{21}\beta_{12}}{1-\gamma_{12}\gamma_{21}}, \quad \Pi_{13} = \frac{\beta_{13}}{1-\gamma_{12}\gamma_{21}}$$

$$\pi_{23} = \frac{\gamma_{21}\beta_{13}}{1-\gamma_{12}\gamma_{21}}$$

note that if we use INDIRECT LEAST SQUARES, WE have 2 CONSISTENT ESTIMATORS OF:

$$\hat{\gamma}_{21} = \frac{\hat{\pi}_{22}}{\hat{\pi}_{12}} \quad \text{and} \quad \hat{\gamma}_{21} = \frac{\hat{\pi}_{23}}{\hat{\pi}_{13}}$$

ILS DO NOT YIELD A UNIQUE $\hat{\gamma}_{21}$ ESTIMATOR: IDENTIFICATION PROBLEM. note that empirically

$$\frac{\hat{\pi}_{22}}{\hat{\pi}_{12}} \neq \frac{\hat{\pi}_{23}}{\pi_{13}}.$$

IDENTIFICATION

$$p_t = \gamma_{12} q_t + \beta_{11} + e_{1t}$$

$$q_t = \gamma_{21} p_t + \beta_{21} + e_{2t}$$

We need to estimate: γ_{12}, γ_{21}, β_{11}, β_{21}, σ_1^2, σ_2^2, σ_{12}. (7 parameters)

reduced form :

$$p_t = \Pi_{10} + v_{1t}$$

$$q_t = \Pi_{20} + v_{2t}$$

estimating both equations of the reduced form by OLS separately, we may obtain estimators of

Π_{10}, Π_{20}, σ_{v_1}, σ_{v_2}, $\sigma_{v_1 v_2}$, 5 parameters

while we need to find 7.

The system is underidentified and there is no estimation method available.

In the contrary situation system is overidentified and 2SLS methods works only

If the system is exactly identified, ILS and 2SLS work.

ORDER CONDITIONS (necessary identification conditions)

⇒ informs if the equation i is over, under or exactly identified

g^* number of endogenous in equation i

K^* exogenous in equation i (and predetermined))

K total number of exogenous in the whole system!

1) when

$$(g^* + K^* - 1) < K$$

or

$$g^* - 1 < K - K^*$$

EQUATION IS OVERIDENTIFIED (2SLS) IV*

2) $g^* + K^* - 1 = K$

$$g^* - 1 = K - K^*$$

EXACTLY IDENTIFIED (ILS, 2SLS, IV)

3*) $g^* + K^* - 1 > K$ or $g^* - 1 > K - K^*$ UNDERIDEN

sufficient condition = rank condition

2SLS

for exactly or overidentified equations:

$$p_t = \beta_{11} + \gamma_{12} q_t + \beta_{12} ps_t + \beta_{13} di_t + e_{1t}$$

$$q_t = \beta_{21} + \gamma_{21} p_t + \beta_{24} pf_t + e_{2t}$$

$$y_1 = \begin{bmatrix} y_2 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix} + e_1 = Z_1 \delta_1 + e_1$$

$$y_2 = \begin{bmatrix} y_1 & x_1 & x_4 \end{bmatrix} \begin{bmatrix} \gamma_{21} \\ \beta_{21} \\ \beta_{24} \end{bmatrix} + e_2 = Z_2 \delta_2 + e_2$$

consider equation 1:

# of endo in equation 1 : 2

# of exo absent in 1 = 4 - 3 = 1 (only x_4 is missing)

$$g^* - 1 = 1 = K - K^* = 1$$

equation 1 is exactly identified

equation 2: $g^* = 2 \Rightarrow g^* - 1 = 1$

$K - K^* = 4 - 2 = 2$

$$g^* - 1 = 1 < K - K^* = 2$$

overidentified.

We estimate equation 2 by 2SLS.

1. Build the reduced form.
   Find the expression for Y_1, the endogenous present in equation 2:

$$Y_1 = x_1 \Pi_{11} + x_2 \Pi_{12} + x_3 \Pi_{13} + x_4 \Pi_{14} + V_1$$

$$= X \Pi_1 + V_1$$

2. Estimate it by OLS

$$Y_1 = X \hat{\Pi}_1 + \hat{V}_1$$

$$Y_2 = X \hat{\Pi}_2 + \hat{V}_2$$

$$\hat{Y}_2 = X \hat{\Pi}_2$$

3. Find fitted Y_1's i.e. $\hat{Y}_1$:

$$\hat{Y}_1 = X \hat{\Pi}_1 \qquad \Rightarrow Y_1 = \hat{Y}_1 + \hat{V}_1$$

$$\hat{V}_1 = \cancel{\ } - \hat{Y}_1 = Y_1 - X \hat{\Pi}_1$$

4.

in expression:

$$Y_2 = Z_2\delta_2 + e_2 = \begin{bmatrix} Y_1 & x_1 & x_4 \end{bmatrix} \begin{bmatrix} \gamma_{21} \\ \beta_{21} \\ \beta_{24} \end{bmatrix} + e_2$$

replace: y_1 by $\hat{y}_1 + \hat{v}_1$

$$Y_2 = Z_2\delta_2 + e_2 = \begin{bmatrix} (\hat{y}_1 + \hat{v}_1) & x_1 & x_4 \end{bmatrix} \begin{bmatrix} \delta_{21} \\ \beta_{21} \\ \beta_{24} \end{bmatrix} + e_2$$

Let $\bar{e}_2 = \hat{v}_1 \gamma_{21} + e_2$

$$Y_2 = \hat{Z}_2\delta_2 + \bar{e}_2 = \begin{bmatrix} \hat{Y}_1 & x_1 & x_4 \end{bmatrix} \begin{bmatrix} \gamma_{21} \\ \beta_{21} \\ \beta_{24} \end{bmatrix} + (e_2 + \hat{v}_1\gamma_{21})$$

- as $T \to \infty$, $\hat{Z}_2$ and $\bar{e}_2$ are uncorrelated. It is not true in small sample as $\hat{Y}_1 = \hat{\Pi}_1 \cdot X$ where $\hat{\Pi}_1 = (X'X)^{-1}X'Y$.

We have a CONSISTENT 2SLS:

$$\hat{\delta}_2^{2SLS} = \left[\hat{Z}_2'\hat{Z}_2\right]^{-1}\hat{Z}_2'y_2$$

with $\hat{var}(\hat{\delta}_2^{2SLS}) = \hat{\sigma}_{22}^2\left[\hat{Z}_2'\hat{Z}_2\right]^{-1}$

where $$\hat{\sigma}_{22}^2 = \frac{(y_2 - \hat{Z}_2\hat{\delta}_2^{2SLS})'(y_2 - \hat{Z}_2\hat{\delta}_2^{2SLS})}{T}$$

We will use the same approach to estimate equation 1:

$$y_1 = \begin{bmatrix} y_2 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix} + e_1$$

$$= Z_1 \delta_1 + e_1$$

replace y_2 by $(x\hat{\pi}_2 + \hat{v}_2) = \hat{y}_2 + \hat{v}_2$

$$y_1 = \begin{bmatrix} \hat{y}_2 + \hat{v}_2 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix} + e_1$$

$$= \begin{bmatrix} \hat{y}_2 & x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} \gamma_{12} \\ \beta_{11} \\ \beta_{12} \\ \beta_{13} \end{bmatrix} + (\gamma_{12} \hat{v}_2 + e_1)$$

$$Y_1 = \hat{Z}_1 \delta_1 + \bar{e}_1$$

the consistent 2SLS for equation 1:

$$\hat{\delta}_1^{2SLS} = (\hat{Z}_1'\hat{Z}_1)^{-1}\hat{Z}_1 Y_1$$

with variance estimated by

$$\hat{var}(\hat{\delta}_1^{2SLS}) = \hat{\sigma}_{11}^2 [\hat{Z}_1'\hat{Z}_1]^{-1}$$

where $\hat{\sigma}_{11}^2 = (Y_1 - \hat{Z}_1\hat{\delta}_1^{2SLS})'(Y_1 - \hat{Z}_1\hat{\delta}_1^{2SLS})/T$