FUNDAMENTAL CONCEPTS

STOCHASTIC PROCESS:

A family of random variables (r.v.) indexed by time.

Notation (Wei):

- stochastic process: $\{Z(t): t = \dots -2, -1, 0, 1, 2 \dots\}$
- components: random variables Z_t

Other text: $\{X(t); t \in \mathbb{Z}\}$, X_t, $\{Y(t); t \in \mathbb{Z}\}$, Y_t

From the stochastic process $\dots Z_{-2}, Z_{-1}, Z_0, Z_1, Z_2 \dots$ we may observe a finite set of random variables $\{Z_{t_1}, Z_{t_2} \dots Z_{t_n}\}$ at times $t_1, t_2 \dots t_n$

- Every component Z_t has its own distribution, density,
  - the mean $E(Z_t)$
  - the variance $var(Z_t)$

- Every pair of components can be characterized by its linear dependence in terms of
  - autocovariance $E(Z_{t_1} - E(Z_{t_1}))(Z_{t_2} - E(Z_{t_2}))$
  - autocorrelation: $\dfrac{\text{autocovariance}(Z_{t_1}, Z_{t_2})}{\sqrt{var(Z_{t_1})}\sqrt{var(Z_{t_2})}}$

STATIONARITY

Strict stationarity: a stochastic process is said to be strictly stationary if all its components have the same distribution and all bivariate, trivariate and other multivariate distributions of its components are invariant with respect to a time shift.

SECOND ORDER STATIONARITY

A TIME SERIES IS "STATIONARY" WHEN

1. The MEAN IS CONSTANT, i.e. does not vary in time:

$$E(z_t) = E(z_{t-1}) = \cdots = \mu$$

2. The VARIANCE IS CONSTANT, i.e. does not vary in time

$$var(z_t) = var(z_{t-1}) = \cdots$$

3. The AUTOCOVARIANCE between two components k periods apart, z_t and z_{t+k} depends on k only (i.e. the distance in time) and not on t (i.e. not on the time)

3 $\Rightarrow$ The AUTOCORRELATION DEPENDS ON K ONLY AND DOES NOT DEPEND ON t.

THE AUTOCORRELATION FUNCTION (ACF)

DEFINES THE TEMPORAL DEPENDENCE (SERIAL CORRELATION) IN A TIMESERIES, AND REPRESENTS THE MEMORY OF T.S.

ACF IS DERIVED FROM THE AUTOCOVARIANCE FUNCTION

1. AUTOCOVARIANCE FUNCTION

Notation: the autocovariance between Z_t and Z_{t+K} is denoted by $\gamma(K)$ or γ_K

It follows that the autocovariance between Z_t and Z_t, itself is the variance of Z_t and is denoted by $\gamma(0)$

Definition:

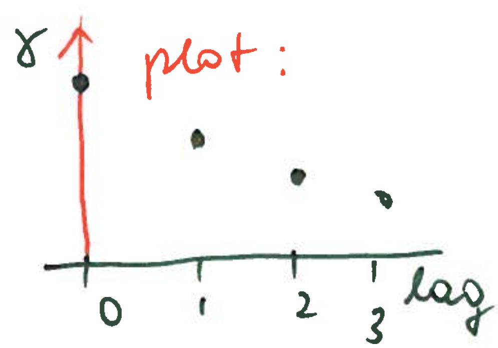
$$\gamma_K = \text{cov}(Z_t, Z_{t+K}) = E(Z_t - \mu)(Z_{t+K} - \mu)$$

It is a function ☺ : the argument is the lag K

$K = \ldots -2, -1, 0, 1, 2 \ldots$ etc. Conventionally, we define γ_K for $K = 0, 1, 2, \ldots$

lag	autocovariance
0	γ_0
1	γ_1
2	γ_2
3	γ_3

plot:



the autocovariance function is an even function, i.e:

$\gamma(-1) = \gamma(1)$, $\gamma(-2) = \gamma(2)$ OR $\gamma_{-1} = \gamma_1$, $\gamma_{-2} = \gamma_2$ etc

also for a time series of length n (sample size $= n$) with components $Z_1, Z_2 \ldots . Z_n$ we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \operatorname{cov}(Z_{t_i}, Z_{t_j}) = \sum \sum \alpha_i \alpha_j \gamma_{|t_i - t_j|}$$

where $t_i - t_j = k$ and appears here in absolute value.

ex: $n=3$ and $\{Z_1, Z_2, Z_3\}$ with common mean $\mu = 0$

$$\operatorname{var}\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = E[Z Z'] = E\left\{\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} [Z_1 \quad Z_2 \quad Z_3]\right\}$$

$$\begin{bmatrix} E Z_1^2 & E(Z_1 Z_2) & E(Z_1 Z_3) \\ E(Z_1 Z_2) & E Z_2^2 & E(Z_2 Z_3) \\ E(Z_1 Z_3) & E(Z_2 Z_3) & E Z_3^3 \end{bmatrix} = \begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_0 & \gamma_1 \\ \gamma_2 & \gamma_1 & \gamma_0 \end{bmatrix} = \Gamma_3$$

let α be a row vector : $\alpha = [\alpha_1 \ \alpha_2 \ \alpha_3]$. of real numbers.

$$0 \leq \operatorname{var}(\alpha Z) = [\alpha_1 \ \alpha_2 \ \alpha_3] \, \Gamma_3 \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}$$

2. AUTOCORRELATION FUNCTION ACF

$$\rho_K = \frac{\gamma_K}{\gamma_0} = \frac{\text{cov}(Z_t, Z_{t+K})}{\sqrt{\text{var}(Z_t)}\sqrt{\text{var}(Z_{t+K})}}$$

Properties:

1. $\rho_0 = 1$ because $\rho_0 = \frac{\gamma_0}{\gamma_0} = 1$

2. $|\rho_K| \leq 1$

3. is an even function: $\rho_K = \rho_{-K}$

4. for real α_i, α_j, $i,j = 1, \ldots n$: $\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \rho_{|t_i - t_j|} \geq 0$

for ex from page 4.

$$\frac{1}{\gamma_0} \Gamma_3 = \begin{bmatrix} \frac{\gamma_0}{\gamma_0} & \frac{\gamma_1}{\gamma_0} & \frac{\gamma_2}{\gamma_0} \\ \frac{\gamma_1}{\gamma_0} & \frac{\gamma_0}{\gamma_0} & \frac{\gamma_1}{\gamma_0} \\ \frac{\gamma_2}{\gamma_0} & \frac{\gamma_1}{\gamma_0} & \frac{\gamma_0}{\gamma_0} \end{bmatrix} = \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{bmatrix}$$

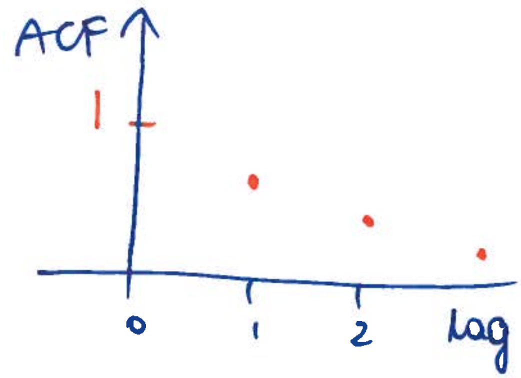
symmetric, positive semidefinite.

The ACF is a function: the argument is the lag K

Display:

lag	autocorrelation
0	1
1	ρ_1
2	ρ_2
3	ρ_3

Plot:



To know about ACF:

- stationary processes from the ARMA (autoregressive moving average) family display short memory: the ACF decreases to 0 at an exponential rate.

- integrated processes, which are nonstationary and display global or local trends have infinite memory: ACF is equal to 1 at any lag.

- white noise processes

  a sequence of independent or uncorrelated random variables with mean 0 (conventional) and constant variance σ^2, denoted by a_t

$$\rho_k = \begin{cases} 1 & \text{at } k=0 \\ 0 & \text{else} \end{cases}$$

ESTIMATION OF MEAN, VARIANCE, and ACF.

IS MEANINGFUL ONLY IF A TIME SERIES IS ERGODIC, i.e. has autocovariances that die out (become = 0) sufficiently fast when the lag K increases.

formally: $\sum_{K=0}^{\infty} |\gamma_K| < \infty$

SUPPOSE THAT WE HAVE A SAMPLE OF SIZE n

→ notation: most texts use T as sample size symbol

- Sample mean $\bar{z} = \frac{1}{n} \sum_{t=1}^{n} z_t$

on page 17, Wei: $\bar{z}$ is shown to be an unbiased and consistent estimator.
Note that $\mathrm{var}(\bar{z}) \to 0$ only if $\rho_K \to 0$ as $K \to \infty$
Hence, the variance of $\bar{z}$ in small sample and/or processes with strong persistence can be arbitrarily large. (see page 18)

means that in practice, estimation of time series requires large samples and is valid in large samples in most cases. (asymptotically valid)

Sample autocovariance

two estimators are available:

$$\hat{\gamma}_K = \frac{1}{n} \sum_{t=1}^{n-K} (z_t - \bar{z})(z_{t+K} - \bar{z})$$

$$\hat{\hat{\gamma}}_K = \frac{1}{n-K} \sum_{t=1}^{n-K} (z_t - \bar{z})(z_{t+K} - \bar{z})$$

both estimators are biased. In general, $\hat{\gamma}_K$ has a larger bias than $\hat{\hat{\gamma}}_K$, especially when K is large with respect to n

Note: for a given sample size n, the number of observations available for estimation of γ_K diminishes when K increases. (ex page 21)

however, $\hat{\gamma}_K$ is always pos. semi-def while $\hat{\hat{\gamma}}_K$ not always.
the biases diminish with the sample size. (page 19)
Both these estimators are used in practice.

- $\hat{\gamma}_0 = \widehat{\operatorname{var}}(z_t) = \frac{1}{n} \sum_{t=1}^{n} (z_t - \bar{z})^2$

SAMPLE ACF:

$$\hat{\rho}_K = \frac{\hat{\gamma}_K}{\hat{\gamma}_0} = \frac{\sum_{t=1}^{n-k} (z_t - \bar{z})(z_{t+k} - \bar{z})}{\sum_{t=1}^{n} (z_t - \bar{z})^2}$$

In many instances we want to test the null hypothesis of autocorrelation being equal to zero at a given lag or for a given set of lags. We say that we test the null hypothesis of a White Noise.

to do that we need a benchmark to assess the statistical significance. For that we need to know the asymptotic distribution of $\hat{\rho}_K$. → $\mathrm{var}(\hat{\rho}_K)$ or $\sqrt{\mathrm{var}\,\hat{\rho}_K}$

- complicated, Bartlett formula for the square root of the variance, called the standard deviation to be used in the confidence interval or test

$$s\hat{\rho}_K = \sqrt{\frac{1}{n}\left(1 + 2\hat{\rho}_1^2 + \cdots + 2\hat{\rho}_m^2\right)}$$

is valid for processes where $\rho_K = 0$ for $K > m$ SAS

- easy, used in other software: We say $\rho_K = 0$ with probability 95% if $\hat{\rho}_k$ is inside the interval

$$\left[-2\frac{1}{\sqrt{n}}\,,\; +2\frac{1}{\sqrt{n}}\right]$$

this assumes asymptotic normality and instead −1.96, +1.
can be used for more precise outcomes.

$$\sqrt{n}\,(\hat{\rho}_K - 0) \overset{asy}{\sim} N(0,1)$$

The LAG OPERATOR

notation Wei: It is denoted by B for "backward"
other texts use L for lag.

$$B\,Z_t = Z_{t-1}$$

$$B^K Z_t = Z_{t-K}$$

$$\frac{1}{B} Z_t = B^{-1} Z_t = Z_{t+1}$$

$$B^{-2} Z_t = Z_{t+2}$$

The Difference operator

notation Wei: $(1-B)$. other texts use Δ or ∇
to denote $\Delta = Z_t - Z_{t-1}$. hence

$$\Delta = Z_t - B\,Z_t = (1-B)\,Z_t$$

so that $\Delta = 1-B$.

alternatively $\Delta = 1-L$ in other texts.

THE AUTOREGRESSIVE PROCESSES

- AR means that Z_t depends on its past values, i.e on Z_{t-1}, Z_{t-2}, etc. Depending on the number of those past Z values, which determine the present Z_t, we distinguish

AR(1) : Z_t depends on Z_{t-1} only

AR(2) : Z_t Z_{t-1} and Z_{t-2}

AR(3) : Z_t Z_{t-1}, Z_{t-2} and Z_{t-3}

or, in general : on **p** past values of Z_t:

AR(p)

- the simplest model is the AR(1)

AR(1) IS DEFINED AS:

$$Z_t = \delta + \phi_1 Z_{t-1} + a_t$$

- a_t IS A WHITE NOISE (SEQUENCE OF UNCORRELATED R.V.); $a_t \sim (0, \sigma_a^2) \Rightarrow E(a_t) = 0$ and $var(a_t) = \sigma_a^2$

$cov(a_t, a_{t+k}) = 0 \quad \forall \, k \neq 0$.

$E(a_t Z_{t-1}) = 0 \Rightarrow cov(a_t, Z_{t-1}) = 0$

- ϕ_1 IS THE AUTOREGRESSIVE COEFFICIENT. ITS VALUE IS CRUCIAL AS IT DETERMINES IF THE PROCESS IS STATIONARY

- δ IS A CONSTANT

AR(1) IS STATIONARY $\Leftrightarrow |\phi_1| < 1$

$\Rightarrow$ to check, we consider the mean, variance and covariance

$$E(z_t) = E[\delta + \phi_1 z_{t-1} + a_t]$$

suppose $E(z_t) = E(z_{t-1}) = \dots = \mu$

$$E(z_t) = \frac{\delta}{1-\phi_1} = \mu$$

FROM NOW ON, FOR CONVENIENCE WE ASSUME THAT THE ORIGINAL TIME SERIES HAS BEEN TRANSFORMED BY REMOVING THE MEAN μ FROM ALL REALIZATIONS $t=1,\dots n$

$$\dot{z}_t = z_t - \frac{\delta}{1-\phi_1} = z_t - \mu$$

and $E(\dot{z}_t) = 0$

$$var(\dot{z}_t) = var(\phi_1 \dot{z}_{t-1} + a_t)$$

$$= \phi_1^2 \, var(\dot{z}_{t-1}) + var(a_t)$$

because $\dot{z}_{t-1}$ and a_t are UNCORRELATED

$$= \phi_1^2 \, var(\dot{z}_t) + \sigma_a^2$$

by stationarity $var(\dot{z}_{t-1}) = var(\dot{z}_t)$

$$var(\dot{z}_t) = \frac{\sigma_a^2}{1-\phi_1^2}$$

autocovariances

$$\gamma_0 = var(\dot{z}_t) = \frac{\sigma_a^2}{1-\phi_1^2}$$

at lag 1:

$$cov(\dot{z}_t, \dot{z}_{t-1}) = E\left[(\dot{z}_t - E(\dot{z}_t))(\dot{z}_{t-1} - E(\dot{z}_{t-1}))\right]$$

$$= E\left[\dot{z}_t \dot{z}_{t-1}\right]$$

$$= E\left[(\phi_1 \dot{z}_{t-1} + a_t)\dot{z}_{t-1}\right] =$$

$$= \phi_1 E(\dot{z}_{t-1}^2) + \underbrace{E(a_t z_{t-1})}_{=0}$$

0 because a_t and $\dot{z}_{t-1}$ uncorrelat

$$\gamma_1 = \phi_1 \gamma_0$$

the same result holds for

$cov(\dot{z}_{t-1}, \dot{z}_{t-2})$, $cov(\dot{z}_{t-5}, \dot{z}_{t-6})$, i.e. all $\dot{z}$ values one period apart

at lag 2:

$$cov(\dot{z}_t, \dot{z}_{t-2}) = E\left[(\dot{z}_t - E(\dot{z}_t))(\dot{z}_{t-2} - E(\dot{z}_{t-2}))\right]$$

$$= E\left[\dot{z}_t \; \dot{z}_{t-2}\right]$$

$$= E\left[(\phi_1 \dot{z}_{t-1} + a_t)\, \dot{z}_{t-2}\right]$$

$$= \phi_1 E(\dot{z}_{t-1} \dot{z}_{t-2}) + \underbrace{E\left[a_t \dot{z}_{t-2}\right]}_{=0}$$

$$\gamma_2 = \phi_1 \cdot \phi_1 \cdot \gamma_0 = \phi_1^2 \gamma_0$$

We can generalize these results for any lag $K = 1, 2 \ldots$

$$\gamma_K = \phi_1^K \gamma_0 = cov(\dot{z}_t, \dot{z}_{t-K})$$

THIS IS THE AUTOCOVARIANCE FUNCTION OF AR(1)

THE AUTOCORRELATION FUNCTION (ACF) OF AR(1)

$$\rho_K = \frac{\gamma_K}{\gamma_0} = \phi_1^K$$

ACF

lag K	ρ_K
0	1
1	ϕ_1
2	ϕ_1^2
3	ϕ_1^3
4	ϕ_1^4

if $\phi_1 = 0.3$ if $\phi_1 = 0.9$

Note that given $|\phi_1| < 1$, the ACF gradually die out to zero. The rate is expo and the pace depends on the value of ϕ_1.

The effect of the initial condition $\dot{Z}_0$ on $\dot{Z}_t$:

$$
\begin{aligned}
\dot{Z}_t &= \phi_1 \dot{Z}_{t-1} + a_t \\
&= \phi_1 (\phi_1 \dot{Z}_{t-2} + a_{t-1}) + a_t \\
&= \phi_1^2 (\phi_1 \dot{Z}_{t-3} + a_{t-2}) + a_t + \phi_1 a_{t-1} \\
&= \phi_1^3 (\phi_1 \dot{Z}_{t-4} + a_{t-3}) + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} \\
&\ \vdots \\
&= \phi_1^t \dot{Z}_0 + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \\
&\quad + \cdots + \phi_1^{t-1} a_1
\end{aligned}
$$

estimation of ACF:

$$\hat{\rho}_1 = \frac{\widehat{\mathrm{cov}}(\dot{z}_t, \dot{z}_{t-1})}{\widehat{\mathrm{var}}\, \dot{z}_t} = \frac{\sum_{t=2}^{n} \dot{z}_t \cdot \dot{z}_{t-1}}{\sum_{t=1}^{n} \dot{z}_t^2}$$

$$\hat{\rho}_2 = \frac{\widehat{\mathrm{cov}}(\dot{z}_t, \dot{z}_{t-2})}{\widehat{\mathrm{var}}\, \dot{z}_t} = \frac{\sum_{t=3}^{n} \dot{z}_t \cdot \dot{z}_{t-2}}{\sum_{t=1}^{n} \dot{z}_t^2}$$

testing for WHITE NOISE:

$H_0: \rho_1 = 0$

under H_0: $\sqrt{n}\,\hat{\rho}_1 \overset{a}{\sim} N(0,1)$

Reject H_0 if $|\hat{\rho}_1| \geq \frac{1.96}{\sqrt{n}}$ or $\frac{2}{\sqrt{n}}$

AR(2):

$$Z_t = \delta + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

$$E(Z_t) = E(Z_{t-1}) = E(Z_{t-2}) = \mu$$

$$\mu = \delta + \phi_1 \mu + \phi_2 \mu$$

$$E(Z_t) = \frac{\delta}{1-\phi_1-\phi_2} = \mu$$

FOR STATIONARITY OF AR(2) WE REQUIRE:

1. $\phi_2 + \phi_1 < 1$
2. $\phi_2 - \phi_1 < 1$
3. $|\phi_2| < 1$

let $\dot{Z}_t = Z_t - \mu$, and recall $B\dot{Z}_t = \dot{Z}_{t-1}$, $B^2\dot{Z}_t = Z_{t-2}$

$$\dot{Z}_t = \phi_1 B \dot{Z}_t + \phi_2 B^2 \dot{Z}_t + a_t$$

$$\dot{Z}_t (1 - \phi_1 B - \phi_2 B^2) = a_t$$

The polynomial in B has roots outside the unit circle, i.e. of modulus greater than 1 if 1., 2., 3. hold.

autocovariances.

$\gamma_0 = E(\dot{z}_t^2)$ is the $var(\dot{z}_t)$

- $\gamma_0 = E\left[(\phi_1 \dot{z}_{t-1} + \phi_2 \dot{z}_{t-2} + a_t)\, \dot{z}_t\right]$

$= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_a^2$

- $\gamma_K = E\left[\dot{z}_{t-K}\, \dot{z}_t\right] = E\left[\dot{z}_{t-K}(\phi_1 \dot{z}_{t-1} + \phi_2 \dot{z}_{t-2} + a_t\right.$

$= \phi_1 \gamma_{K-1} + \phi_2 \gamma_{K-2}$

ACF of AR(2):

- $\rho_K = \phi_1 \rho_{K-1} + \phi_2 \rho_{K-2}$ for $K > 2$

and $\rho_1 = \dfrac{\phi_1}{1-\phi_2}$, $\rho_2 = \phi_2 + \dfrac{\phi_1^2}{1-\phi_2}$