THE AUTOREGRESSIVE PROCESSES

- AR means that Z_t depends on its past values, i.e on Z_{t-1}, Z_{t-2}, etc. Depending on the number of those past Z values, which determine the present Z_t, we distinguish

AR(1) : Z_t depends on Z_{t-1} only

AR(2) : Z_t Z_{t-1} and Z_{t-2}

AR(3) : Z_t Z_{t-1}, Z_{t-2} and Z_{t-3}

or, in general : on **p** past values of Z_t:

AR(p)

- the simplest model is the AR(1)

AR(1) IS DEFINED AS:

$$Z_t = \delta + \phi_1 Z_{t-1} + a_t$$

- a_t IS A WHITE NOISE (SEQUENCE OF UNCORRELATED R.V.); $a_t \sim (0, \sigma_a^2) \Rightarrow E(a_t) = 0$ and $var(a_t) = \sigma_a^2$

$cov(a_t, a_{t+k}) = 0 \quad \forall\ k \neq 0$.

$E(a_t Z_{t-1}) = 0 \Rightarrow cov(a_t, Z_{t-1}) = 0$

- ϕ_1 IS THE AUTOREGRESSIVE COEFFICIENT. ITS VALUE IS CRUCIAL AS IT DETERMINES IF THE PROCESS IS STATIONARY

- δ IS A CONSTANT

AR(1) IS STATIONARY $\iff |\phi_1| < 1$

$\Rightarrow$ to check, we consider the mean, variance and covariance

$$E(z_t) = E[\delta + \phi_1 z_{t-1} + a_t]$$

suppose $E(z_t) = E(z_{t-1}) = \dots = \mu$

$$E(z_t) = \frac{\delta}{1-\phi_1} = \mu$$

FROM NOW ON, FOR CONVENIENCE WE ASSUME THAT THE ORIGINAL TIME SERIES HAS BEEN TRANSFORMED BY REMOVING THE MEAN μ FROM ALL REALIZATIONS $t=1,\dots n$

$$\dot{z}_t = z_t - \frac{\delta}{1-\phi_1} = z_t - \mu$$

and $E(\dot{z}_t) = 0$

$$\mathrm{var}(\dot{z}_t) = \mathrm{var}(\phi_1 \dot{z}_{t-1} + a_t)$$

$$= \phi_1^2 \, \mathrm{var}(\dot{z}_{t-1}) + \mathrm{var}(a_t)$$

because $\dot{z}_{t-1}$ and a_t are UNCORRELATED

$$= \phi_1^2 \, \mathrm{var}(\dot{z}_t) + \sigma_a^2$$

by stationarity $\mathrm{var}(\dot{z}_{t-1}) = \mathrm{var}(\dot{z}_t)$

$$\mathrm{var}(\dot{z}_t) = \frac{\sigma_a^2}{1-\phi_1^2}$$

autocovariances

$$\gamma_0 = \mathrm{var}(\dot{z}_t) = \frac{\sigma_a^2}{1-\phi_1^2}$$

at lag 1 :

$$\mathrm{cov}(\dot{z}_t, \dot{z}_{t-1}) = E\left[(\dot{z}_t - E(\dot{z}_t))(\dot{z}_{t-1} - E(\dot{z}_{t-1}))\right]$$

$$= E\left[\dot{z}_t \dot{z}_{t-1}\right]$$

$$= E\left[(\phi_1 \dot{z}_{t-1} + a_t)\dot{z}_{t-1}\right] =$$

$$= \phi_1 E(\dot{z}_{t-1}^2) + \underbrace{E(a_t z_{t-1})}_{= 0}$$

0 because a_t and $\dot{z}_{t-1}$ uncorrelat

$$\gamma_1 = \phi_1 \gamma_0$$

the same result holds for

$cov(\dot{z}_{t-1}, \dot{z}_{t-2})$, $cov(\dot{z}_{t-5}, \dot{z}_{t-6})$, i.e. all $\dot{z}$ values one period apart

at lag 2:

$$cov(\dot{z}_t, \dot{z}_{t-2}) = E\left[(\dot{z}_t - E(\dot{z}_t))(\dot{z}_{t-2} - E(\dot{z}_{t-2}))\right]$$

$$= E\left[\dot{z}_t \ \dot{z}_{t-2}\right]$$

$$= E\left[(\phi_1 \dot{z}_{t-1} + a_t)\ \dot{z}_{t-2}\right]$$

$$= \phi_1 E(\dot{z}_{t-1} \dot{z}_{t-2}) + \underbrace{E\left[a_t \dot{z}_{t-2}\right]}_{=0}$$

$$\gamma_2 = \phi_1 \cdot \phi_1 \cdot \gamma_0 = \phi_1^2 \gamma_0$$

We can generalize these results for any lag $K = 1, 2 \ldots$

$$\gamma_K = \phi_1^K \gamma_0 = cov(\dot{z}_t, \dot{z}_{t-K})$$

THIS IS THE AUTOCOVARIANCE FUNCTION OF AR(1)

THE AUTOCORRELATION FUNCTION (ACF) OF AR(1)

$$\rho_K = \frac{\gamma_K}{\gamma_0} = \phi_1^K$$

ACF

lag K	ρ_K
0	1
1	ϕ_1
2	ϕ_1^2
3	ϕ_1^3
4	ϕ_1^4

if $\phi_1 = 0.3$ if $\phi_1 = 0.9$

Note that given $|\phi_1| < 1$, the ACF gradually die out to zero. The rate is expo and the pace depends on the value of ϕ_1.

The effect of the initial condition $\dot{Z}_0$ on $\dot{Z}_t$:

$$\dot{Z}_t = \phi_1 \dot{Z}_{t-1} + a_t$$

$$= \phi_1 (\phi_1 \dot{Z}_{t-2} + a_{t-1}) + a_t$$

$$= \phi_1^2 (\phi_1 \dot{Z}_{t-3} + a_{t-2}) + a_t + \phi_1 a_{t-1}$$

$$= \phi_1^3 (\phi_1 \dot{Z}_{t-4} + a_{t-3}) + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2}$$

$$\vdots$$

$$= \phi_1^t \dot{Z}_0 + a_t + \phi_1 a_{t-1} + \phi_1^2 a_{t-2} + \dots + \phi_1^{t-1} a_1$$

estimation of ACF:

$$\hat{\rho}_1 = \frac{\hat{cov}(\dot{z}_t, \dot{z}_{t-1})}{\hat{var}\, \dot{z}_t} = \frac{\sum_{t=2}^{n} \dot{z}_t \cdot \dot{z}_{t-1}}{\sum_{t=1}^{n} \dot{z}_t^2}$$

$$\hat{\rho}_2 = \frac{\hat{cov}(\dot{z}_t, z_{t-2})}{\hat{var}\, \dot{z}_t} = \frac{\sum_{t=3}^{n} \dot{z}_t \cdot \dot{z}_{t-2}}{\sum_{t=1}^{n} \dot{z}_t^2}$$

testing for WHITE NOISE:

$H_0: \rho_1 = 0$

under H_0: $\sqrt{n}\, \hat{\rho}_1 \overset{a}{\sim} N(0,1)$

Reject H_0 if $|\hat{\rho}_1| \geq \frac{1.96}{\sqrt{n}}$ or $\frac{2}{\sqrt{n}}$

AR(2):

$$Z_t = \delta + \phi_1 Z_{t-1} + \phi_2 Z_{t-2} + a_t$$

$$E(Z_t) = E(Z_{t-1}) = E(Z_{t-2}) = \mu$$

$$\mu = \delta + \phi_1 \mu + \phi_2 \mu$$

$$E(Z_t) = \frac{\delta}{1-\phi_1-\phi_2} = \mu$$

FOR STATIONARITY OF AR(2) WE REQUIRE:

1. $\phi_2 + \phi_1 < 1$
2. $\phi_2 - \phi_1 < 1$
3. $|\phi_2| < 1$

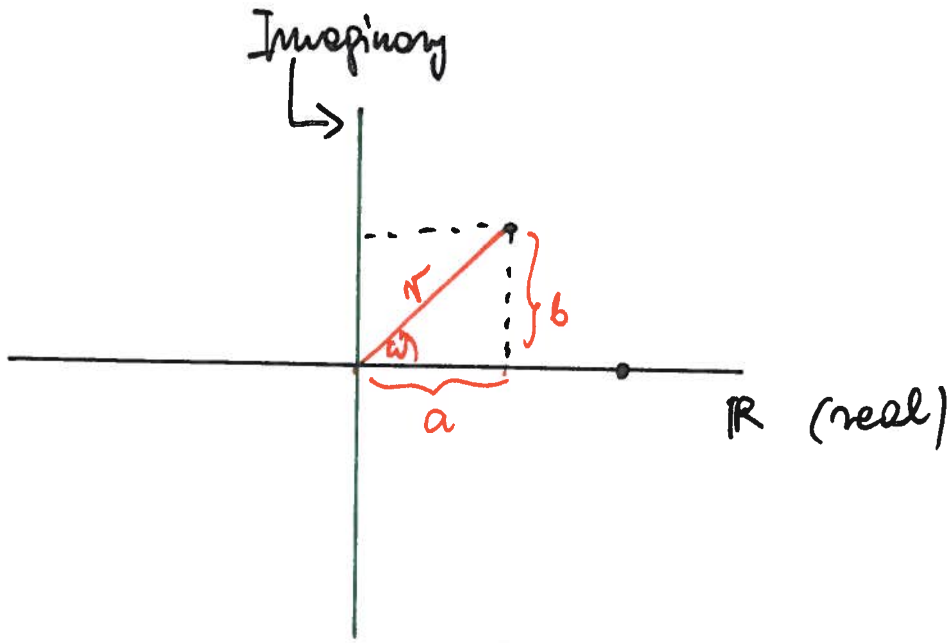
let $\dot{Z}_t = Z_t - \mu$, and recall $B\dot{Z}_t = \dot{Z}_{t-1}$, $B^2\dot{Z}_t = Z_{t-2}$

$$\dot{Z}_t = \phi_1 B\dot{Z}_t + \phi_2 B^2\dot{Z}_t + a_t$$

$$\dot{Z}_t(1 - \phi_1 B - \phi_2 B^2) = a_t$$

The polynomial in B has roots outside the unit circle, i.e. of modulus greater than 1 if 1., 2., 3. hold.

$i = \sqrt{-1}$



the modulus of a real number = absolute value

modulus of an imaginary # $a \pm bi \Rightarrow r = \sqrt{a^2+b^2}$

AR(2) has an autoregressive polynomial of order 2 in B:

$$1 - \phi_1 B - \phi_2 B^2 = 0$$

or $$-\phi_2 B^2 - \phi_1 B + 1 = 0$$

which has two (or one) roots B_1, B_2 either real or imaginary

$$B_i = \frac{\phi_1 \pm \sqrt{\Delta}}{-2\phi_2}, \quad i = 1, 2. \quad , \text{depending on } \Delta = \phi_1^2 + 4\phi_2$$

being greater, equal or less than zero.

conditions 1, 2 $\Rightarrow$ if roots B_1, B_2 real, then greater >1 in absolute value

condition 3 $\Rightarrow$ if roots imaginary $a+bi = r\cos\omega + ri\sin\omega$ then the oscillations are damped if greater >1 IN MODULUS autocovariances.

$\gamma_0 = E(\dot{z}_t^2)$ is the $var(\dot{z}_t)$

- $\gamma_0 = E[(\phi_1 \dot{z}_{t-1} + \phi_2 \dot{z}_{t-2} + a_t)\dot{z}_t]$

  $= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_a^2$

- $\gamma_K = E[\dot{z}_{t-K} \dot{z}_t] = E[\dot{z}_{t-K}(\phi_1 \dot{z}_{t-1} + \phi_2 \dot{z}_{t-2} + a_t$

  $= \phi_1 \gamma_{K-1} + \phi_2 \gamma_{K-2}$

ACF of AR(2):

- $\rho_K = \phi_1 \rho_{K-1} + \phi_2 \rho_{K-2}$ for $K > 2$

and $\rho_1 = \dfrac{\phi_1}{1-\phi_2}$, $\rho_2 = \phi_2 + \dfrac{\phi_1^2}{1-\phi_2}$

AR(p) : STATIONARITY: The roots of the autoregressive polynomial have to lie outside the unit circle.

AR(p) have the autocovariances and ACF defined by

Yule-Walker equations

$$\gamma_K = \phi_1 \gamma_{K-1} + \cdots + \phi_p \gamma_{K-p} \quad K>0$$

$$\rho_K = \phi_1 \rho_{K-1} + \cdots + \phi_p \rho_{K-p} \quad K>0$$