

Chapter 13

VEC and VAR Models: An Introduction to Macroeconometrics

Learning Objectives

Based on the material in this chapter, you should be able to do the following:

1. Explain why economic variables are dynamically interdependent.
2. Explain the VEC model.
3. Explain the importance of error correction.
4. Explain the VAR model.
5. Explain the relationship between a VEC model and a VAR model.
6. Explain how to estimate the VEC and VAR models for the bivariate case.
7. Explain how to generate impulse response functions and variance decompositions for the simple case when the variables are not contemporaneously interdependent and when the shocks are not correlated.

Keywords

dynamic relationships
error correction
forecast error variance decomposition
identification problem

impulse response functions
VAR model
VEC model

In Chapter 12, we studied the time-series properties of data and cointegrating relationships between pairs of nonstationary series. In those examples, we assumed that one of the variables was the dependent variable (let us call it y_t) and the other was the independent variable (say x_t), and we treated the relationship between y_t and x_t like a regression model. However, a priori, unless we have good reasons not to, we could just as easily have assumed that y_t is the independent variable and x_t is the dependent variable. Put simply, we are working with two variables $\{y_t, x_t\}$ and the two possible regression models relating them are

$$y_t = \beta_{10} + \beta_{11}x_t + e_t^y, \quad e_t^y \sim N(0, \sigma_y^2) \quad (13.1a)$$

$$x_t = \beta_{20} + \beta_{21}y_t + e_t^x, \quad e_t^x \sim N(0, \sigma_x^2) \tag{13.1b}$$

In this bivariate (two series) system there can be only one unique relationship between x_t and y_t , and so it must be the case that $\beta_{21} = 1/\beta_{11}$ and $\beta_{20} = -\beta_{10}/\beta_{11}$. A bit of terminology: for (13.1a) we say that we have normalized on y (meaning the coefficient in front of y is set to 1) whereas for (13.1b) we say that we have normalized on x (meaning the coefficient in front of x is set to 1).

Is it better to write the relationship as (13.1a) or (13.1b), or is it better to recognize that, in many relationships, variables like y and x are simultaneously determined? The aim of this chapter is to explore the causal relationship between pairs of time-series variables. In doing so, we shall be extending our study of time-series data to take account of their dynamic properties and interactions. In particular, we will discuss the **vector error correction (VEC)** and **vector autoregressive (VAR)** models. We will learn how to estimate a VEC model when there is cointegration between $I(1)$ variables, and how to estimate a VAR model when there is no cointegration.

Some important terminology emerges here. Univariate analysis examines a single data series. Bivariate analysis examines a pair of series. The term **vector** indicates that we are considering a number of series, two, three, or more. The term "vector" is a generalization of the univariate and bivariate cases.

13.1 VEC and VAR Models

Let us begin with two time-series variables y_t and x_t and generalize the discussion about dynamic relationships in Chapter 9 to yield a system of equations:

$$\begin{aligned} y_t &= \beta_{10} + \beta_{11}y_{t-1} + \beta_{12}x_{t-1} + v_t^y \\ x_t &= \beta_{20} + \beta_{21}y_{t-1} + \beta_{22}x_{t-1} + v_t^x \end{aligned} \tag{13.2}$$

The equations in (13.2) describe a system in which each variable is a function of its own lag, and the lag of the other variable in the system. In this case, the system contains two variables y and x . In the first equation y_t is a function of its own lag y_{t-1} and the lag of the other variable in the system x_{t-1} . In the second equation x_t is a function of its own lag x_{t-1} and the lag of the other variable in the system y_{t-1} . Together the equations constitute a system known as a vector autoregression (VAR). In this example, since the maximum lag is of order 1, we have a VAR(1).

If y and x are stationary $I(0)$ variables, the above system can be estimated using least squares applied to each equation. If, however, y and x are nonstationary $I(1)$ and not cointegrated, then, as discussed in Chapter 12, we work with the first differences. In this case, the VAR model is

$$\begin{aligned} \Delta y_t &= \beta_{11}\Delta y_{t-1} + \beta_{12}\Delta x_{t-1} + v_t^{\Delta y} \\ \Delta x_t &= \beta_{21}\Delta y_{t-1} + \beta_{22}\Delta x_{t-1} + v_t^{\Delta x} \end{aligned} \tag{13.3}$$

All variables are now $I(0)$, and the system can again be estimated by least squares. To recap: the VAR model is a general framework to describe the dynamic interrelationship between stationary variables. Thus, if y and x are stationary $I(0)$ variables, the system in equation (13.2) is used. On the other hand, if y and x are $I(1)$ variables but they are not cointegrated, we examine the interrelation between them using a VAR framework in first differences (13.3).

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If y and x are $I(1)$ and cointegrated, then we need to modify the system of equations to allow for the cointegrating relationship between the $I(1)$ variables. We do this for two reasons. First, as economists, we like to retain and use valuable information about the cointegrating relationship and second, as econometricians, we like to ensure that we use the best technique that takes into account the properties of the time-series data. Recall the chapter on simultaneous equations—the cointegrating equation is one way of introducing simultaneous interactions without requiring the data to be stationary. Introducing the cointegrating relationship leads to a model known as the VEC model. We turn now to this model.

Consider two nonstationary variables y_t and x_t that are integrated of order 1; $y_t \sim I(1)$ and $x_t \sim I(1)$ and which we have shown to be cointegrated, so that

$$y_t = \beta_0 + \beta_1 x_t + e_t \quad (13.4)$$

Note that we could have chosen to normalize on x . Whether we normalize on y or x is often determined from economic theory; the critical point is that there can be at most one fundamental relationship between the two variables.

The VEC model is a special form of the VAR for $I(1)$ variables that are cointegrated. The VEC model is

$$\begin{aligned} \Delta y_t &= \alpha_{10} + \alpha_{11}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^y \\ \Delta x_t &= \alpha_{20} + \alpha_{21}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^x \end{aligned} \quad (13.5a)$$

which we can expand as

$$\begin{aligned} y_t &= \alpha_{10} + (\alpha_{11} + 1)y_{t-1} - \alpha_{11}\beta_0 - \alpha_{11}\beta_1 x_{t-1} + v_t^y \\ x_t &= \alpha_{20} + \alpha_{21}y_{t-1} - \alpha_{21}\beta_0 - (\alpha_{21}\beta_1 - 1)x_{t-1} + v_t^x \end{aligned} \quad (13.5b)$$

Comparing (13.5b) with (13.2) shows the VEC as a VAR where the $I(1)$ variable y_t is related to other lagged variables (y_{t-1} and x_{t-1}) and where the $I(1)$ variable x_t is also related to the other lagged variables (y_{t-1} and x_{t-1}). Note, however, that the two equations contain the common cointegrating relationship.

The coefficients α_{11} , α_{21} are known as error correction coefficients, so named because they show how much Δy_t and Δx_t respond to the cointegrating error $y_{t-1} - \beta_0 - \beta_1 x_{t-1} = e_{t-1}$. The idea that the error leads to a correction comes about because of the conditions put on α_{11} , α_{21} to ensure stability, namely $(-1 < \alpha_{11} \leq 0)$ and $(0 \leq \alpha_{21} < 1)$. To appreciate this idea, consider a positive error $e_{t-1} > 0$ that occurred because $y_{t-1} > (\beta_0 + \beta_1 x_{t-1})$. A negative error correction coefficient in the first equation (α_{11}) ensures that Δy falls, while the positive error correction coefficient in the second equation (α_{21}) ensures that Δx rises, thereby correcting the error. Having the error correction coefficients less than 1 in absolute value ensures that the system is not explosive.

The error correction model has become an extremely popular model because its interpretation is intuitively appealing. Think about two nonstationary variables, say consumption (let us call it y_t) and income (let us call it x_t), that we expect to be related (cointegrated). Now think about a change in your income Δx_t , say a pay raise! Consumption will most likely increase, but it may take you a while to change your consumption pattern in response to a change in your pay. The VEC model allows us to examine how much consumption will change in response to a change in the explanatory variable (the cointegration part, $y_t = \beta_0 + \beta_1 x_t + e_t$), as well as the speed of the change (the error correction part, $\Delta y_t = \alpha_{10} + \alpha_{11}(e_{t-1}) + v_t^y$ where e_{t-1} is the cointegrating error).

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$$\begin{matrix} 1 + v_t^y \\ 1 + v_t^x \end{matrix} \quad (13.5b)$$

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There is one final point to discuss—the role of the intercept terms. Thus far, we have introduced an intercept term in the cointegrating equation (β_0) as well as in the VEC (α_{10} and α_{20}). However, doing so can create a problem. To see why, we collect all the intercept terms and rewrite (13.5b) as

$$\begin{aligned} y_t &= (\alpha_{10} - \alpha_{11}\beta_0) + (\alpha_{11} + 1)y_{t-1} - \alpha_{11}\beta_1 x_{t-1} + v_t^y \\ x_t &= (\alpha_{20} - \alpha_{21}\beta_0) + \alpha_{21}y_{t-1} - (\alpha_{21}\beta_1 - 1)x_{t-1} + v_t^x \end{aligned} \quad (13.5c)$$

If we estimate each equation by least squares, we obtain estimates of composite terms ($\alpha_{10} - \alpha_{11}\beta_0$) and ($\alpha_{20} - \alpha_{21}\beta_0$), and we are not able to disentangle the separate effects of β_0 , α_{10} , and α_{20} . In the next section, we discuss a simple two-step least squares procedure that gets around this problem. However, the lesson here is to check whether and where an intercept term is needed.

13.2 Estimating a Vector Error Correction Model

There are many econometric methods to estimate the error correction model. The most straightforward is to use a two-step least squares procedure. First, use least squares to estimate the cointegrating relationship $y_t = \beta_0 + \beta_1 x_t + e_t$ and generate the lagged residuals $\hat{e}_{t-1} = y_{t-1} - b_0 - b_1 x_{t-1}$.

Second, use least squares to estimate the equations:

$$\Delta y_t = \alpha_{10} + \alpha_{11} \hat{e}_{t-1} + v_t^y \quad (13.6a)$$

$$\Delta x_t = \alpha_{20} + \alpha_{21} \hat{e}_{t-1} + v_t^x \quad (13.6b)$$

Note that all the variables in equation (13.6) (Δy , Δx and \hat{e}) are stationary (recall that for y and x to be cointegrated, the residuals \hat{e} must be stationary). Hence, the standard regression analysis studied in earlier chapters may be used to test the significance of the parameters. The usual residual diagnostic tests may be applied.

We need to be careful here about how we combine stationary and nonstationary variables in a regression model. Cointegration is about the relationship between $I(1)$ variables. The cointegrating equation does not contain $I(0)$ variables. The corresponding VEC model, however, relates the change in an $I(1)$ variable (the $I(0)$ variables Δy and Δx) to other $I(0)$ variables, namely the cointegration residuals \hat{e}_{t-1} and, if required, other stationary variables may be added. In other words, we should not mix stationary and nonstationary variables: an $I(0)$ dependent variable on the left-hand side of a regression equation should be “explained” by other $I(0)$ variables on the right-hand side and an $I(1)$ dependent variable on the left-hand side of a regression equation should be “explained” by other $I(1)$ variables on the right-hand side.

13.2.1 EXAMPLE

In Figure 13.1 the quarterly real GDP of a small economy (Australia) and a large economy (United States) for the sample period 1970.1 to 2000.4 are displayed. Note that the series have been scaled so that both economies show a real GDP value of 100 in 2000. They appear in the file *gdp.dat*. It appears from the figure that both series are nonstationary and possibly cointegrated.

Formal unit root tests of the series confirm that they are indeed nonstationary. To check for cointegration we obtain the fitted equation in (13.7) (the intercept term is omitted

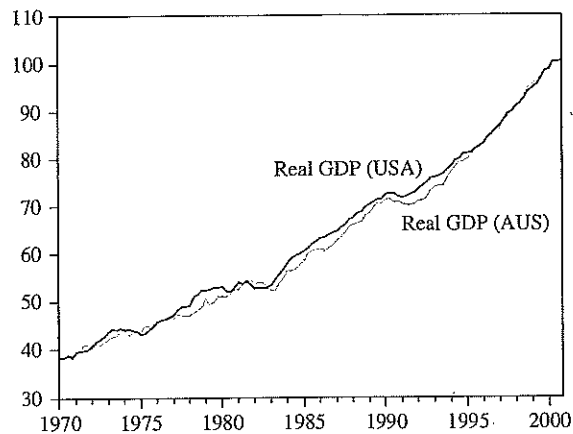


FIGURE 13.1 Real gross domestic products (GDP).

because it has no economic meaning):

$$\hat{A}_t = 0.985U_t, \quad R^2 = 0.995 \quad (13.7)$$

where A denotes real GDP for Australia and U denotes real GDP for the United States. Note that we have normalized on A because it makes more sense to think of a small economy responding to a large economy. We then performed a test for stationarity of the residuals $\hat{e}_t = A_t - 0.985U_t$. The estimated unit root test equation for the residuals is

$$\begin{aligned} \widehat{\Delta e}_t &= -0.128\hat{e}_{t-1} \\ (\text{tau}) & \quad (-2.889) \end{aligned} \quad (13.8)$$

Since the cointegrating relationship does not contain an intercept term [see Chapter 12, equation (12.8a)], the 5% critical value is -2.76 . The unit root t -value of -2.889 is less than -2.76 . We reject the null of no cointegration and we conclude that the two real GDP series are cointegrated. This result implies that economic activity in the small economy (Australia, A_t) is linked to economic activity in the large economy (United States, U_t). If U_t were to increase by one unit, A_t would increase by 0.985. But the Australian economy may not respond fully by this amount within the quarter. To ascertain how much it will respond within a quarter, we estimate the error correction model by least squares. The estimated VEC model for $\{A_t, U_t\}$ is

$$\begin{aligned} \widehat{\Delta A}_t &= 0.492 - 0.099\hat{e}_{t-1} \\ (t) & \quad (2.077) \\ \widehat{\Delta U}_t &= 0.510 + 0.030\hat{e}_{t-1} \\ (t) & \quad (0.789) \end{aligned} \quad (13.9)$$

The results show that both error correction coefficients are of the appropriate sign. The negative error correction coefficient in the first equation (-0.099) indicates that ΔA falls, while the positive error correction coefficient in the second equation (0.030) indicates that ΔU rises, when there is a positive cointegrating error: ($\hat{e}_{t-1} > 0$ or $A_{t-1} > 0.985U_{t-1}$).

This behavior (negative change in A and positive change in U) "corrects" the cointegrating error. The error correction coefficient (-0.099) is significant at the 5% level; it indicates that the quarterly adjustment of A_t will be about 10% of the deviation of A_{t-1} from its cointegrating value $0.985U_{t-1}$. This is a slow rate of adjustment. However, the error correction coefficient in the second equation (0.030) is insignificant; it suggests that ΔU does not react to the cointegrating error. This outcome is consistent with the view that the small economy is likely to react to economic conditions in the large economy, but not vice versa.

13.3 Estimating a VAR Model

The VEC is a multivariate dynamic model that incorporates a cointegrating equation. It is relevant when, for the two variable case, we have two variables, say y and x , that are both $I(1)$, but are cointegrated. Now we ask: what should we do if we are interested in the interdependencies between y and x , but they are not cointegrated? In this case, we estimate a vector autoregressive (VAR) model as shown in (13.3).

As an example, consider Figure 13.2 that shows the log of GDP (denoted as G) and log of the CPI (denoted as P) for the US economy over the period 1960:1 to 2004:4. The data are in the file *growth.dat*.

The fitted least squares regression of G_t on P_t is

$$\hat{G}_t = 1.632 + 0.624P_t$$

(t) (41.49) (61.482)

For this fitted model $R^2 = 0.955$ and the t -statistics are very large, a seemingly strong result. Based on the plots the series appear to be nonstationary. To test for cointegration, compute the least squares residual $\hat{e}_t = G_t - 1.632 - 0.624P_t$. The Dickey-Fuller regression is

$$\widehat{\Delta e}_t = -0.009\hat{e}_{t-1}$$

(tau) (-0.977) (13.10)

The potential cointegrating relationship contains an intercept term to capture the component of GDP that is independent of the CPI. Thus, the 5% critical value of the test for stationarity

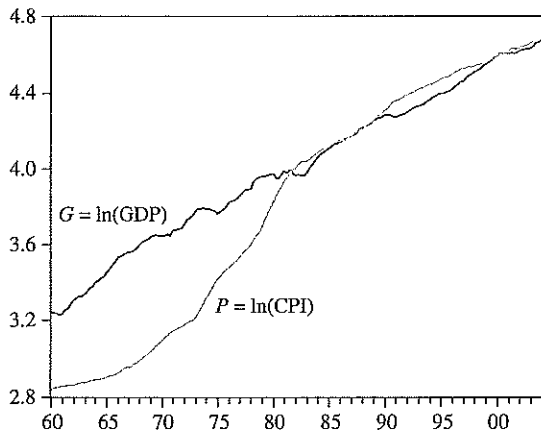


FIGURE 13.2 Real GDP and the consumer price index (CPI) in logarithms.

(13.7)

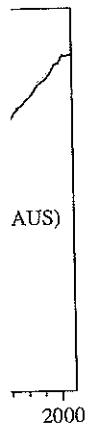
for the United States. Note that the stationarity of the residuals is

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in the cointegrating residuals is -3.37 [see Chapter 12, equation (12.8b)]. Since the tau (unit root t -value) of -0.977 is greater than -3.37 , it indicates that the errors are nonstationary and hence that the relationship between G (i.e., $\ln(GDP)$) and P (i.e., $\ln(CPI)$) is spurious. That is, we have no cointegration. Thus we would not apply a VEC model to examine the dynamic relationship between aggregate price P and output G . Instead we estimate a VAR model for the set of $I(0)$ variables $\{\Delta P_t, \Delta G_t\}$.

For illustrative purposes, the order of lag in this example has been restricted to 1. In general, one should test for the significance of lag terms greater than 1. The results are

$$\widehat{\Delta P_t} = 0.001 + 0.827\Delta P_{t-1} + 0.046\Delta G_{t-1} \quad (13.11a)$$

(t) (2.017)(18.494) (1.165)

$$\widehat{\Delta G_t} = 0.010 - 0.327\Delta P_{t-1} + 0.228\Delta G_{t-1} \quad (13.11b)$$

(t) (7.845)(-4.153) (3.256)

The first equation (13.11a) shows that the quarterly growth in price (ΔP_t) is significantly related to its own past value (ΔP_{t-1}) but insignificantly related to the quarterly growth in last period's GDP (ΔG_{t-1}). The second equation (13.11b) shows that ΔG_t is significantly positively related to its own past value and significantly negatively related to last period's change in price (i.e., inflation). The constant terms capture the fixed component in the change in log price (which is a measure of inflation) and the change in $\ln(GDP)$ (which is a measure of the change in economic activity, or growth in the economy).

Having estimated these models can we infer anything else? If the system is subjected to a price (demand) shock, what is the effect of the shock on the dynamic path of inflation and growth? Will inflation rise and by how much? If the system is also subjected to a quantity (supply) shock, what is the contribution of a price versus a quantity shock on the variation of output? We turn now to some analysis suited to addressing these questions.

13.4 Impulse Responses and Variance Decompositions

Impulse response functions and variance decompositions are techniques that are used by macroeconometricians to analyze problems such as the effect of an oil price shock on inflation and GDP growth, and the effect of a change in monetary policy on the economy.

13.4.1 IMPULSE RESPONSE FUNCTIONS

Impulse response functions show the effects of shocks on the adjustment path of the variables. To help us understand this we shall first consider a univariate series.

13.4.1a The Univariate Case

Consider a univariate series $y_t = \rho y_{t-1} + v_t$ and subject it to a shock of size v in period 1. Assume an arbitrary starting value of y at time zero: $y_0 = 0$. Since we are interested in the dynamic path, the starting point is irrelevant. At time $t = 1$, following the shock, the value of y will be: $y_1 = \rho y_0 + v_1 = v$. Assume that there are no subsequent shocks in later time periods [$v_2 = v_3 = \dots = 0$], at time $t = 2$, $y_2 = \rho y_1 = \rho v$. At time $t = 3$, $y_3 = \rho y_2 = \rho(\rho y_1) = \rho^2 v$, and so on. Thus the time-path of y following the shock is $\{v, \rho v, \rho^2 v, \dots\}$. The values of the coefficients $\{1, \rho, \rho^2, \dots\}$ are known as multipliers, and the time-path of y following the shock is known as the impulse response function.

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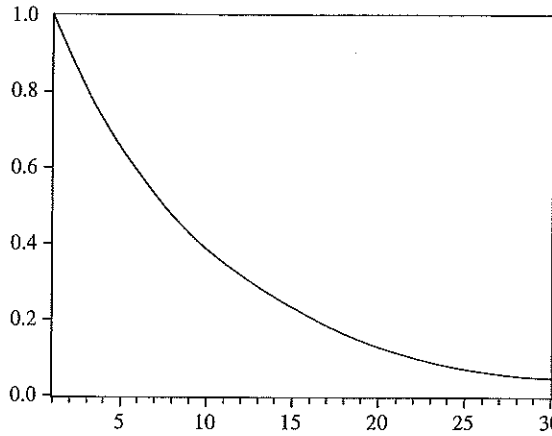


FIGURE 13.3 Impulse responses for an AR(1) model $y_t = 0.9y_{t-1} + e_t$ following a unit shock.

To illustrate, assume that $\rho = 0.9$ and let the shock be unity: $v = 1$. According to the analysis, y will be $\{1, 0.9, 0.81, \dots\}$, approaching zero over time. This impulse response function is plotted in Figure 13.3. It shows us what happens to y after a shock. In this case, y initially rises by the full amount of the shock and then it gradually returns to the value before the shock.

13.4.1b The Bivariate Case

Now, let us consider an impulse response function analysis with two time series based on a bivariate VAR system of stationary variables:

$$\begin{aligned} y_t &= \delta_{10} + \delta_{11}y_{t-1} + \delta_{12}x_{t-1} + v_t^y \\ x_t &= \delta_{20} + \delta_{21}y_{t-1} + \delta_{22}x_{t-1} + v_t^x \end{aligned} \tag{13.12}$$

In this case, there are two possible shocks to the system—one to y and the other to x . Thus we are interested in four impulse response functions—the effect of a shock to y on the time-paths of y and x and the effect of a shock to x on the time-paths of y and x .

The actual mechanics of generating impulse responses in a system is complicated by (i) the fact that one has to allow for interdependent dynamics (the multivariate analog of generating the multipliers) and (ii) one has to identify the correct shock from unobservable data. Taken together, these two complications lead to what is known as the **identification problem**. In this chapter, we consider a special case where there is no identification problem.¹ This special case occurs when the system described in (13.12) is a true representation of the dynamic system—namely, y is related only to lags of y and x , and x is related only to lags of y and x . In other words, y and x are related in a dynamic fashion, but not contemporaneously. The current value x_t does not appear in the equation for y_t , and the current value y_t does not appear in the equation for x_t . Also, we need to assume the errors v_t^y and v_t^x are independent of each other (contemporaneously uncorrelated). In addition, we assume $v^y \sim N(0, \sigma_y^2)$ and $v^x \sim N(0, \sigma_x^2)$.

¹ Appendix 13A introduces the general problem.

Consider the case when there is a one standard deviation shock (alternatively called an **innovation**) to y so that at time $t = 1$, $v_1^y = \sigma_y$, and v_t^y is zero thereafter. Assume $v_t^x = 0$ for all t . It is traditional to consider a standard deviation shock (innovation) rather than a unit shock to overcome measurement issues. Assume $y_0 = x_0 = 0$. Also, since we are focusing on how a shock *changes* the paths of y and x , we can ignore the intercepts. Then

1. When $t = 1$, the effect of a shock of size σ_y on y is $y_1 = v_1^y = \sigma_y$, and the effect on x is $x_1 = v_1^x = 0$.
2. When $t = 2$, the effect of the shock on y is

$$y_2 = \delta_{11}y_1 + \delta_{12}x_1 = \delta_{11}\sigma_y + \delta_{12}0 = \delta_{11}\sigma_y$$

and the effect on x is

$$x_2 = \delta_{21}y_1 + \delta_{22}x_1 = \delta_{21}\sigma_y + \delta_{22}0 = \delta_{21}\sigma_y.$$

3. When $t = 3$, the effect of the shock on y is

$$y_3 = \delta_{11}y_2 + \delta_{12}x_2 = \delta_{11}\delta_{11}\sigma_y + \delta_{12}\delta_{21}\sigma_y$$

and the effect on x is

$$x_3 = \delta_{21}y_2 + \delta_{22}x_2 = \delta_{21}\delta_{11}\sigma_y + \delta_{22}\delta_{21}\sigma_y.$$

By repeating the substitutions for $t = 4, 5, \dots$, we obtain the impulse response of the shock (or innovation) to y on y as $\sigma_y\{1, \delta_{11}, (\delta_{11}\delta_{11} + \delta_{12}\delta_{21}), \dots\}$ and the impulse response of a shock to y on x as $\sigma_y\{0, \delta_{21}, (\delta_{21}\delta_{11} + \delta_{22}\delta_{21}), \dots\}$.

Now consider what happens when there is a one standard deviation shock to x so that at time $t = 1$, $v_1^x = \sigma_x$, and v_t^x is zero thereafter. Assume $v_t^y = 0$ for all t . In the first period after the shock, the effect of a shock of size σ_x on y is $y_1 = v_1^y = 0$, and the effect of the shock on x is $x_1 = v_1^x = \sigma_x$. Two periods after the shock, when $t = 2$, the effect on y is

$$y_2 = \delta_{11}y_1 + \delta_{12}x_1 = \delta_{11}0 + \delta_{12}\sigma_x = \delta_{12}\sigma_x$$

and the effect on x is

$$x_2 = \delta_{21}y_1 + \delta_{22}x_1 = \delta_{21}0 + \delta_{22}\sigma_x = \delta_{22}\sigma_x$$

Again, by repeated substitutions, we obtain the impulse response of a shock to x on y as $\sigma_x\{0, \delta_{12}, (\delta_{11}\delta_{12} + \delta_{12}\delta_{22}), \dots\}$, and the impulse response of a shock to x on x as $\sigma_x\{1, \delta_{22}, (\delta_{21}\delta_{12} + \delta_{22}\delta_{22}), \dots\}$. Figure 13.4 shows the four impulse response functions for numerical values: $\sigma_y = 1$, $\sigma_x = 2$, $\delta_{11} = 0.7$, $\delta_{12} = 0.2$, $\delta_{21} = 0.3$ and $\delta_{22} = 0.6$.

The advantage of examining impulse response functions (and not just VAR coefficients) is that they show the size of the impact of the shock plus the rate at which the shock dissipates, allowing for interdependencies.

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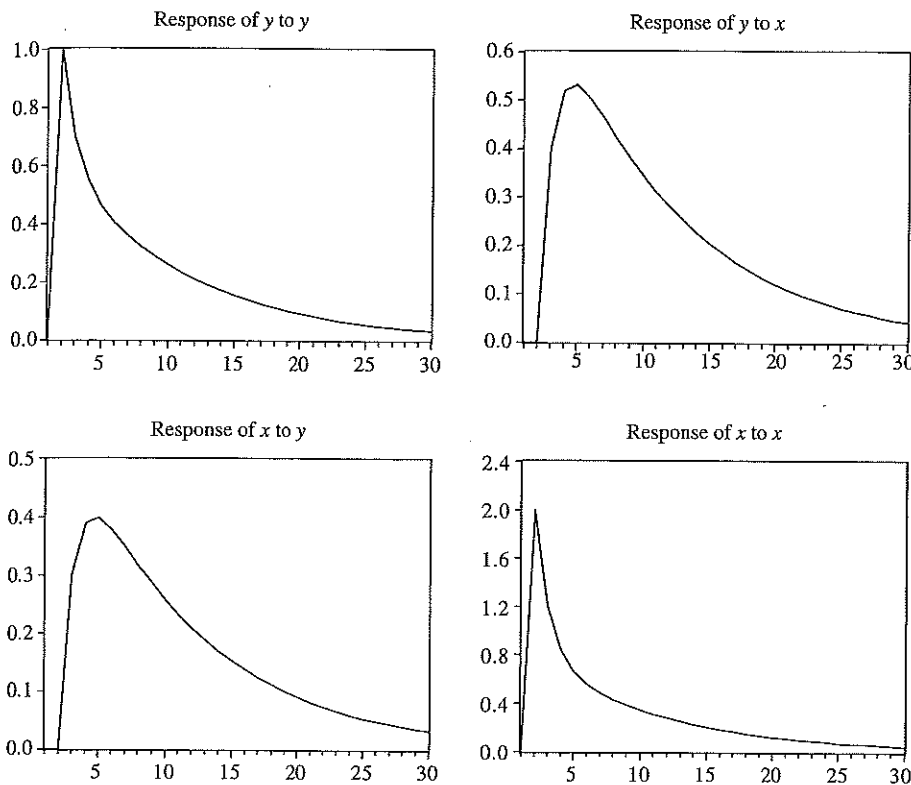


FIGURE 13.4 Impulse responses to standard deviation shock.

13.4.2 FORECAST ERROR VARIANCE DECOMPOSITIONS

Another way to disentangle the effects of various shocks is to consider the contribution of each type of shock to the forecast error variance.

13.4.2a Univariate Analysis

Consider again the univariate series, $y_t = \rho y_{t-1} + v_t$. The best one-step ahead forecast (alternatively the forecast one period ahead) is

$$y_{t+1}^F = E_t[\rho y_t + v_{t+1}]$$

where E_t is the expected value conditional on information at time t (i.e., we are interested in the mean value of y_{t+1} using what is known at time t). At time t the conditional expectation $E_t[\rho y_t] = \rho y_t$ is known, but the error v_{t+1} is unknown, and so its conditional expectation is zero. Thus, the best forecast of y_{t+1} is ρy_t and the forecast error is

$$y_{t+1} - E_t[y_{t+1}] = y_{t+1} - \rho y_t = v_{t+1}$$

The variance of the one-step forecast error is $\text{var}(v_{t+1}) = \sigma^2$. Suppose we wish to forecast two steps ahead, then, using the same logic, the two-step forecast becomes

$$y_{t+2}^F = E_t[\rho y_{t+1} + v_{t+2}] = E_t[\rho(\rho y_t + v_{t+1}) + v_{t+2}] = \rho^2 y_t$$

and the two-step forecast error becomes

$$y_{t+2} - E_t[y_{t+2}] = y_{t+2} - \rho^2 y_t = \rho v_{t+1} + v_{t+2}$$

In this case, the variance of the forecast error is $\text{var}(\rho v_{t+1} + v_{t+2}) = \sigma^2(\rho^2 + 1)$ showing that the variance of forecast error increases as we increase the forecast horizon.

In this univariate example, there is only one shock that leads to a forecast error. Hence the forecast error variance is 100% due to its own shock. The exercise of attributing the source of the variation in the forecast error is known as variance decomposition.

13.4.2b Bivariate Analysis

We can perform a variance decomposition for our special bivariate example where there is no identification problem. Ignoring the intercepts (since they are constants), the one-step ahead forecasts are

$$\begin{aligned} y_{t+1}^F &= E_t[\delta_{11}y_t + \delta_{12}x_t + v_{t+1}^y] = \delta_{11}y_t + \delta_{12}x_t \\ x_{t+1}^F &= E_t[\delta_{21}y_t + \delta_{22}x_t + v_{t+1}^x] = \delta_{21}y_t + \delta_{22}x_t \end{aligned}$$

The corresponding one-step ahead forecast errors and variances are

$$\begin{aligned} FE_1^y &= y_{t+1} - E_t[y_{t+1}] = v_{t+1}^y & \text{var}(FE_1^y) &= \sigma_y^2 \\ FE_1^x &= x_{t+1} - E_t[x_{t+1}] = v_{t+1}^x & \text{var}(FE_1^x) &= \sigma_x^2 \end{aligned}$$

Hence in the first period, all variation in the forecast error for y is due to its own shock. Likewise, 100% of the forecast error for x can be explained by its own shock. Using the same technique, the two-step ahead forecast for y is

$$\begin{aligned} y_{t+2}^F &= E_t[\delta_{11}y_{t+1} + \delta_{12}x_{t+1} + v_{t+2}^y] \\ &= E_t[\delta_{11}(\delta_{11}y_t + \delta_{12}x_t + v_{t+1}^y) + \delta_{12}(\delta_{21}y_t + \delta_{22}x_t + v_{t+1}^x) + v_{t+2}^y] \\ &= \delta_{11}(\delta_{11}y_t + \delta_{12}x_t) + \delta_{12}(\delta_{21}y_t + \delta_{22}x_t) \end{aligned}$$

and that for x is

$$\begin{aligned} x_{t+2}^F &= E_t[\delta_{21}y_{t+1} + \delta_{22}x_{t+1} + v_{t+2}^x] \\ &= E_t[\delta_{21}(\delta_{11}y_t + \delta_{12}x_t + v_{t+1}^y) + \delta_{22}(\delta_{21}y_t + \delta_{22}x_t + v_{t+1}^x) + v_{t+2}^x] \\ &= \delta_{21}(\delta_{11}y_t + \delta_{12}x_t) + \delta_{22}(\delta_{21}y_t + \delta_{22}x_t) \end{aligned}$$

The corresponding two-step ahead forecast errors and variances are (recall that we are working with the special case of independent errors)

$$\begin{aligned} FE_2^y &= y_{t+2} - E_t[y_{t+2}] = [\delta_{11}v_{t+1}^y + \delta_{12}v_{t+1}^x + v_{t+2}^y] \\ \text{var}(FE_2^y) &= \delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2 \\ FE_2^x &= x_{t+2} - E_t[x_{t+2}] = [\delta_{21}v_{t+1}^y + \delta_{22}v_{t+1}^x + v_{t+2}^x] \\ \text{var}(FE_2^x) &= \delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2 \end{aligned}$$

We can decompose the total variance of the forecast error for y , $(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2)$, into that due to shocks to y , $(\delta_{11}^2\sigma_y^2 + \sigma_y^2)$, and that due to shocks to x , $(\delta_{12}^2\sigma_x^2)$. This

decomposition is often expressed in proportional terms. The proportion of the two-step forecast error variance of y explained by its "own" shock is

$$(\delta_{11}^2\sigma_y^2 + \sigma_y^2)/(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2)$$

and the proportion of the two-step forecast error variance of y explained by the "other" shock is

$$(\delta_{12}^2\sigma_x^2)/(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2)$$

Similarly, the proportion of the two-step forecast error variance of x explained by its "own" shock is

$$(\delta_{22}^2\sigma_x^2 + \sigma_x^2)/(\delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2)$$

and the proportion of the forecast error of x explained by the "other" shock is

$$(\delta_{21}^2\sigma_y^2)/(\delta_{21}^2\sigma_y^2 + \delta_{22}^2\sigma_x^2 + \sigma_x^2)$$

For our numerical example with $\sigma_y = 1$, $\sigma_x = 2$, $\delta_{11} = 0.7$, $\delta_{12} = 0.2$, $\delta_{21} = 0.3$, and $\delta_{22} = 0.6$, we find that 90.303% of the two-step forecast error variance of y is due to y , and only 9.697% is due to x .

To sum up, suppose you were interested in the relationship between economic growth and inflation. A VAR model will tell you whether they are significantly related to each other; an impulse response analysis will show how growth and inflation react dynamically to shocks, and a variance decomposition analysis will be informative about the sources of volatility.

13.4.2c The General Case

The example above assumes that x and y are not contemporaneously related and that the shocks are uncorrelated. There is no identification problem, and the generation and interpretation of the impulse response functions and decomposition of the forecast error variance are straightforward. In general, this is unlikely to be the case. Contemporaneous interactions and correlated errors complicate the identification of the nature of shocks and hence the interpretation of the impulses and decomposition of the causes of the forecast error variance. This topic is discussed in greater detail in textbooks devoted to time-series analysis.² A description of how the identification problem can arise is given in Appendix 13A.

13.5 Exercises

13.5.1 PROBLEMS

13.1 Consider the following first-order VAR model of stationary variables.

$$\begin{aligned} y_t &= \delta_{11}y_{t-1} + \delta_{12}x_{t-1} + v_t^y \\ x_t &= \delta_{21}y_{t-1} + \delta_{22}x_{t-1} + v_t^x \end{aligned}$$

Under the assumption that there is no contemporaneous dependence, determine the impulse responses, four periods after a standard deviation shock for

² One reference you might consider is Lütkepohl, H. (2005) *Introduction to Multiple Time Series Analysis*, Springer, Chapter 9.

$z_t) = \sigma^2(\rho^2 + 1)$ showing
forecast horizon.
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its own shock. Using the

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$x_t + v_{t+1}^x + v_{t+2}^x]$

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$+ v_{t+2}^y]$

$+ v_{t+2}^x]$

or y , $(\delta_{11}^2\sigma_y^2 + \delta_{12}^2\sigma_x^2 + \sigma_y^2)$,
shocks to x , $(\delta_{12}^2\sigma_x^2)$. This

- (a) y following a shock to y
 (b) y following a shock to x
 (c) x following a shock to y
 (d) x following a shock to x
- 13.2 Consider the first-order VAR model in Exercise 13.1. Under the assumption that there is no contemporaneous dependence determine
- (a) the contribution of a shock to y on the variance of the three-step ahead forecast error for y ;
 (b) the contribution of a shock to x on the variance of the three-step ahead forecast error for y ;
 (c) the contribution of a shock to y on the variance of the three-step ahead forecast error for x ;
 (d) the contribution of a shock to x on the variance of the three-step ahead forecast error for x .
- 13.3 The VEC model is a special form of the VAR for I(1) variables that are cointegrated. Consider the following VEC model:

$$\Delta y_t = \alpha_{10} + \alpha_{11}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^y$$

$$\Delta x_t = \alpha_{20} + \alpha_{21}(y_{t-1} - \beta_0 - \beta_1 x_{t-1}) + v_t^x$$

The VEC model may also be rewritten as a VAR, but the two equations will contain common parameters:

$$y_t = \alpha_{10} + (\alpha_{11} + 1)y_{t-1} - \alpha_{11}\beta_0 - \alpha_{11}\beta_1 x_{t-1} + v_t^y$$

$$x_t = \alpha_{20} + \alpha_{21}y_{t-1} - \alpha_{21}\beta_0 - (\alpha_{21}\beta_1 - 1)x_{t-1} + v_t^x$$

- (a) Suppose you were given the following results of an estimated VEC model.

$$\widehat{\Delta y_t} = 2 - 0.5(y_{t-1} - 1 - 0.7x_{t-1})$$

$$\widehat{\Delta x_t} = 3 + 0.3(y_{t-1} - 1 - 0.7x_{t-1})$$

Rewrite the model in the VAR form.

- (b) Now suppose you were given the following results of an estimated VAR model, but you were also told that y and x are cointegrated.

$$\hat{y}_t = 0.7y_{t-1} + 0.3 + 0.24x_{t-1}$$

$$\hat{x}_t = 0.6y_{t-1} - 0.6 + 0.52x_{t-1}$$

Rewrite the model in the VEC form.

13.5.2 COMPUTER EXERCISES

- 13.4 The data file *gdp.dat* contains quarterly data on the real GDP of Australia (*AUS*) and real GDP of the United States (*USA*) for the sample period 1970.1 to 2000.4.
- (a) Are the series stationary or nonstationary?
 (b) Test for cointegration allowing for an intercept term. You will find that the intercept is negative. Is this sensible? If not, repeat the test for cointegration excluding the constant term.
 (c) Save the cointegrating residuals and estimate the VEC model.