INTRODUCTION

TIME SERIES

MOST OF THE STATISTICAL METHODS APPLY TO OUTCOMES OF INDEPENDENT EXPERIMENTS (RANDOM SAMPLING - CROSS SECTION) WHERE THE ORDER OF OBS. IS IRRELEVANT

ex: consumption of households

here: DATA CONSIST OF OBS ON A R.V. RECORDED AT DIFFERENT TIMES

OBS ARE NOT INDEPENDENT

- A TIME SERIES IS A SEQUENCE OF OBS $\{X_t ; t \in T\}$ INDEXED BY THE SYMBOL t, WHERE $t \in$ to SOME GIVEN ORDERED SET T (THE TIME).

- IF t TAKES A CONTINUOUS RANGE OF REAL VALUES FINITE OR INFINITE, SO THAT $T \subseteq R$, $\{X_t\}$ IS SAID TO BE CONTINUOUS

- IF t TAKES A DISCRETE SET OF VALUES, TYPICALLY $t = 0, \pm 1, \pm 2, \ldots$, THEN $\{X_t\}$ IS SAID TO BE DISCRETE

- TWO TYPES OF DISCRETE T.S. :

THEY ARE DISTINGUISHED ACCORDING TO OBS BEING

INSTANTLY RECORDED : LEVEL (ex: prices, stocks)

RECORDED OVER A TIME INTERVAL : FLOW (income, consumtn)

IN THIS CASE IT IS IMPORTANT TO CONSIDER THE SAMPLING INTERVAL:

$$t = 0, 1, \dots, T/h \equiv N \quad \text{, where}$$

N: SPAN OF THE DATA

h: SAMPLING INTERVAL

T: # OF OBS

IN THIS COURSE WE WILL NOT SPECIFY h EXPLICITELY, WE CONSIDER IT FIXED. WITHOUT LOSS OF GENERALITY, SET $h = 1$, SO THAT $T = N$. WHEN CONSIDERING THE ASYMPTOTIC ANALYSIS IN THIS FRAMEWORK $T \to \infty \Rightarrow N \to \infty$ AT THE SAME RATE.

- EXAMPLES:

1.1. GNP: clear trend, fluctuations around

1.2. GNP: growth rate (same series differenced), no trend, constant mean

1.3. rate of unemployment: no trend, less volatile from 1945 on

1.4. price level: strong upward trend, regime switch 1950

1.5. growth rate of price level: no trend, const. mean, regime switch 1950 - less volatile

3

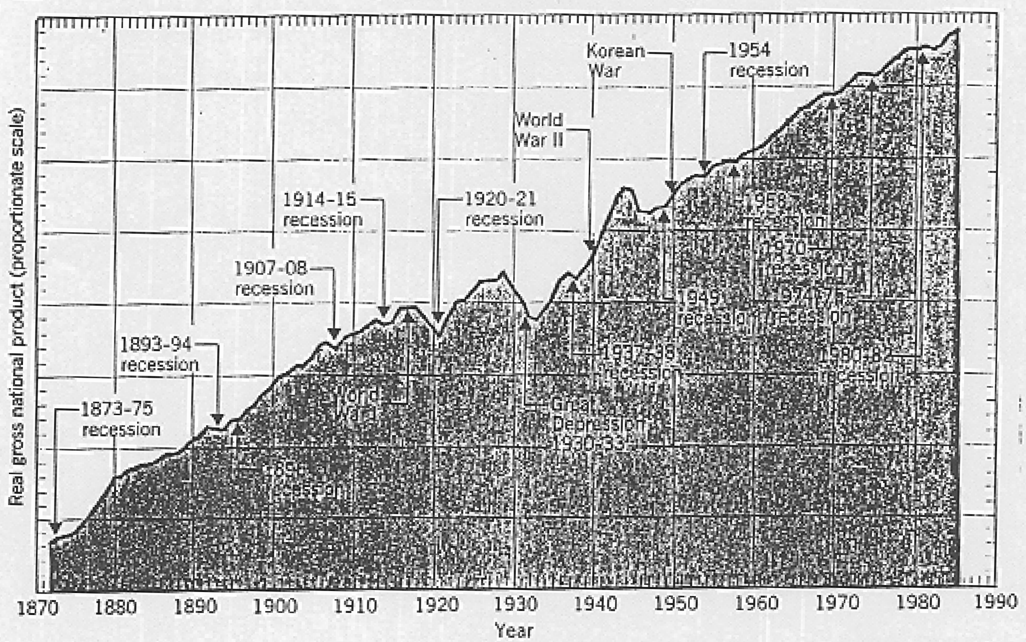


Figure 1.1 The Behavior of Output in the United States, 1872–1985

Sources for Figures 1.1–1.5:

For real GNP—Recent values are from the U.S. Commerce Department, *U.S. Survey of Current Business*. Figures back to 1918 are from the U.S. Commerce Department, *National Income and Product Accounts of the U.S.*, 1929–76. For 1872–1917, the values are from Christina Romer, "The Prewar Business Cycle Reconsidered: New Estimates of Gross National Product, 1872–1918," Princeton University, May 1985, Table 1.

For the GNP deflator—Sources as above since 1918. For 1909–17, figures are from the U.S. Commerce Department, *National Income and Product Accounts of the U.S., 1929–76*. For 1889–1908, the numbers are based on John Kendrick, *Productivity Trends in the United States*, Princeton University Press, Princeton, N.J., 1961, Tables A-1 and A-III. For 1869–88, the data are unpublished estimates of Robert Gallman.

For the unemployment rate—The figures are the number unemployed divided by the total labor force, which includes military personnel. Data since 1930 are from *Economic Report of the President*, 1985, Table B-29; 1983, Table B-29; 1970, Table C-22. The data from 1933–43 are adjusted to classify federal emergency workers as employed, as discussed in Michael Darby, "Three-and-a-Half Million U.S. Employees Have been Mislaid: Or, an Explanation of Unemployment, 1934–1941," *Journal of Political Economy*, February 1976. Values for 1890–1929 are based on Christina Romer, "Spurious Volatility in Historical Unemployment Data," *Journal of Political Economy*, February 1986, Table 9.

upward trend

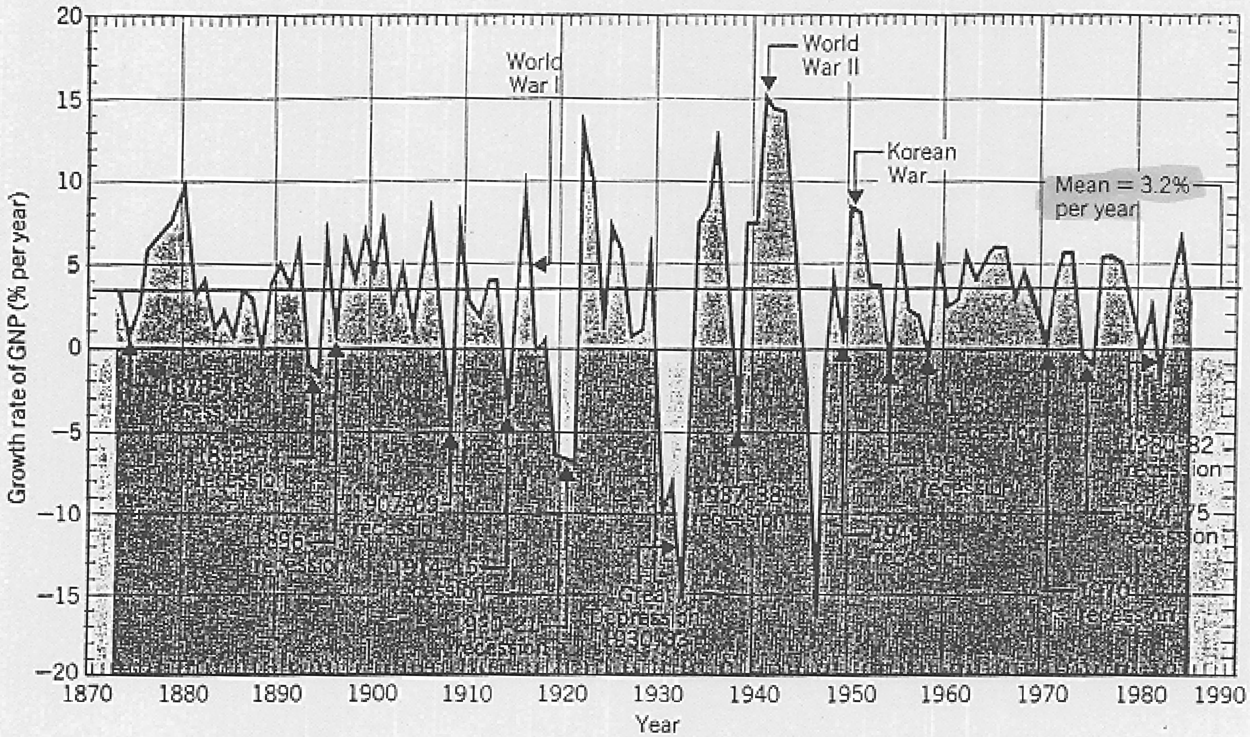


Figure 1.2 Growth Rates of Output in the United States, 1873–1985

differenced series

no trend

const mean

$$\frac{GNP_t - GNP_{t-1}}{GNP_{t-1}} \quad (\%)$$

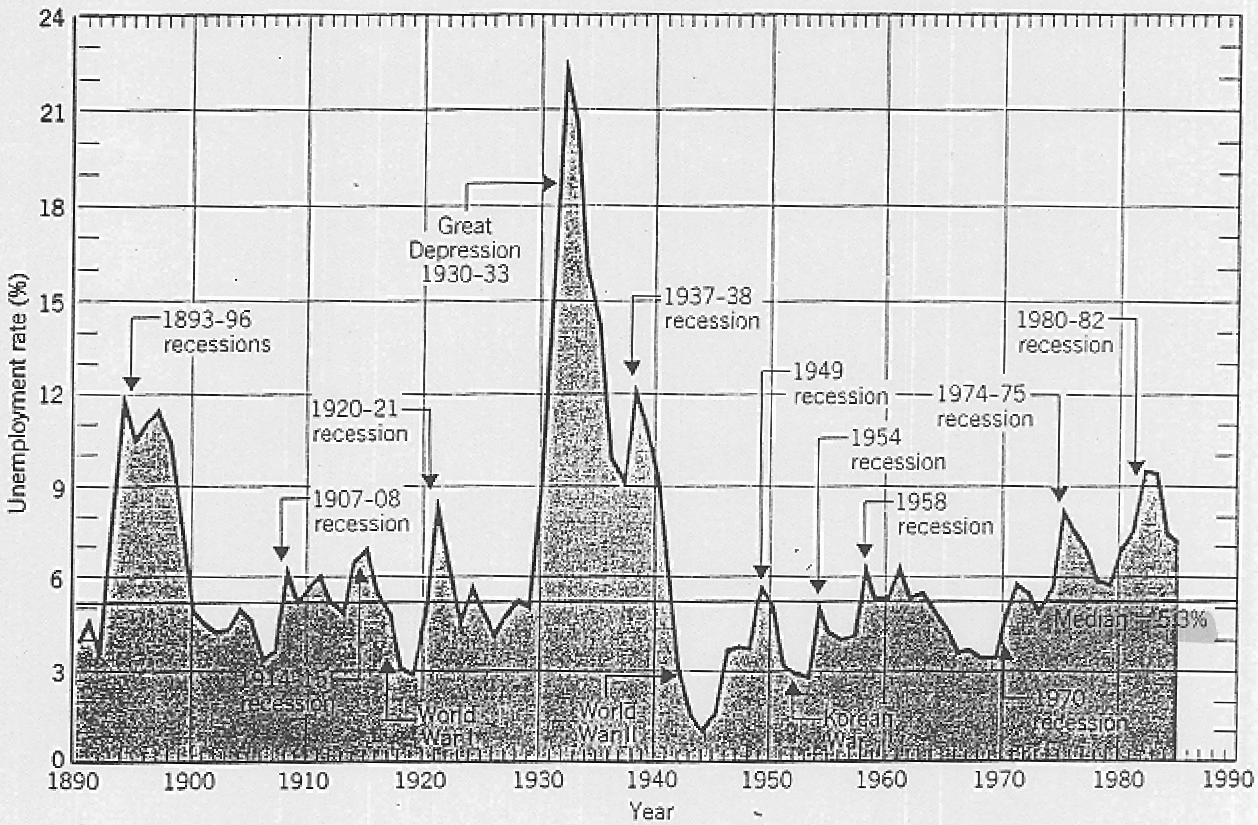


Figure 1.3 The United States Unemployment Rate, 1890–1985

rate (!) of unemployment
less volatile after 1945

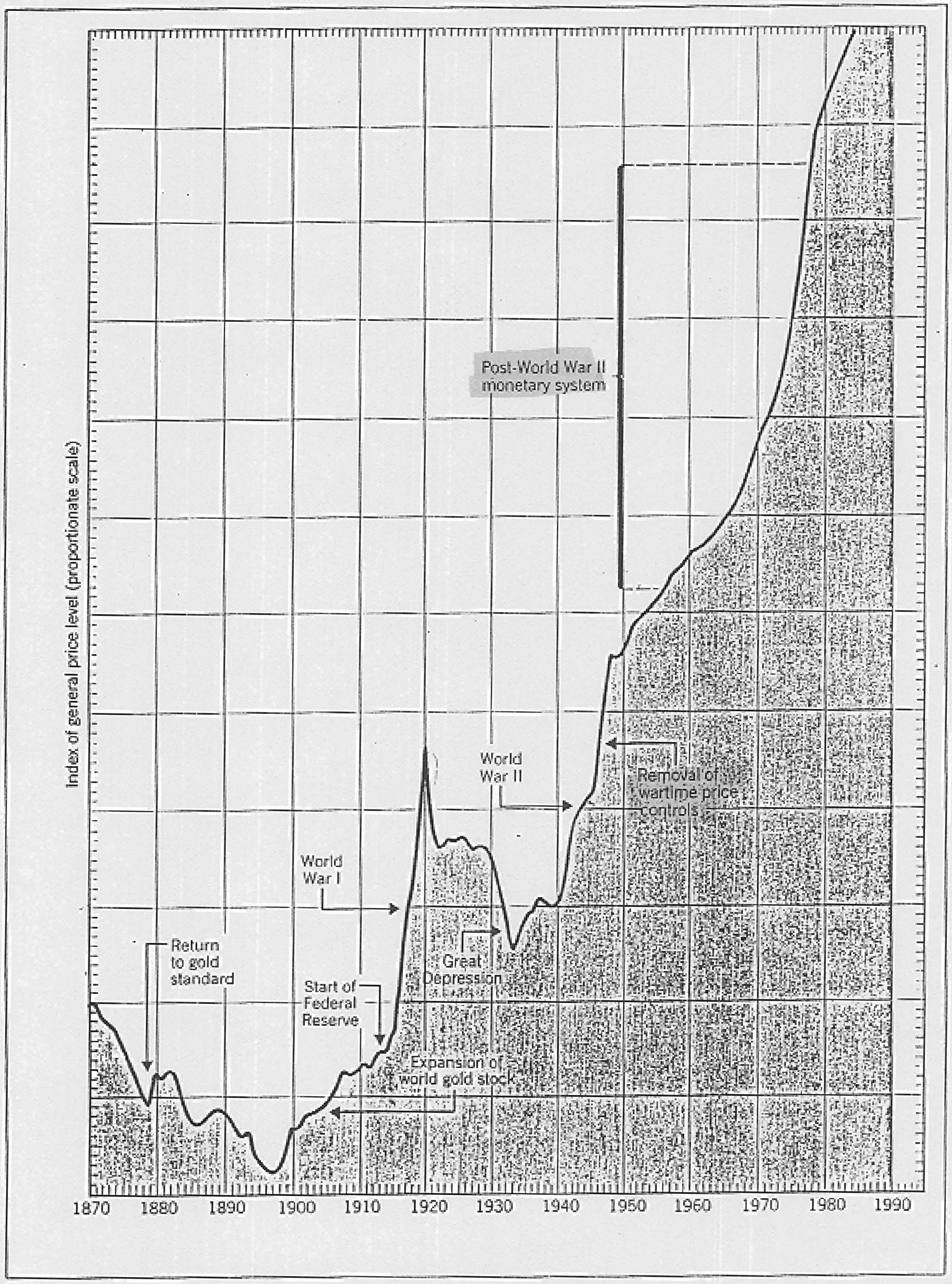


Figure 1.4 The Price Level in the United States, 1870–1985

trend
regime switch 1950

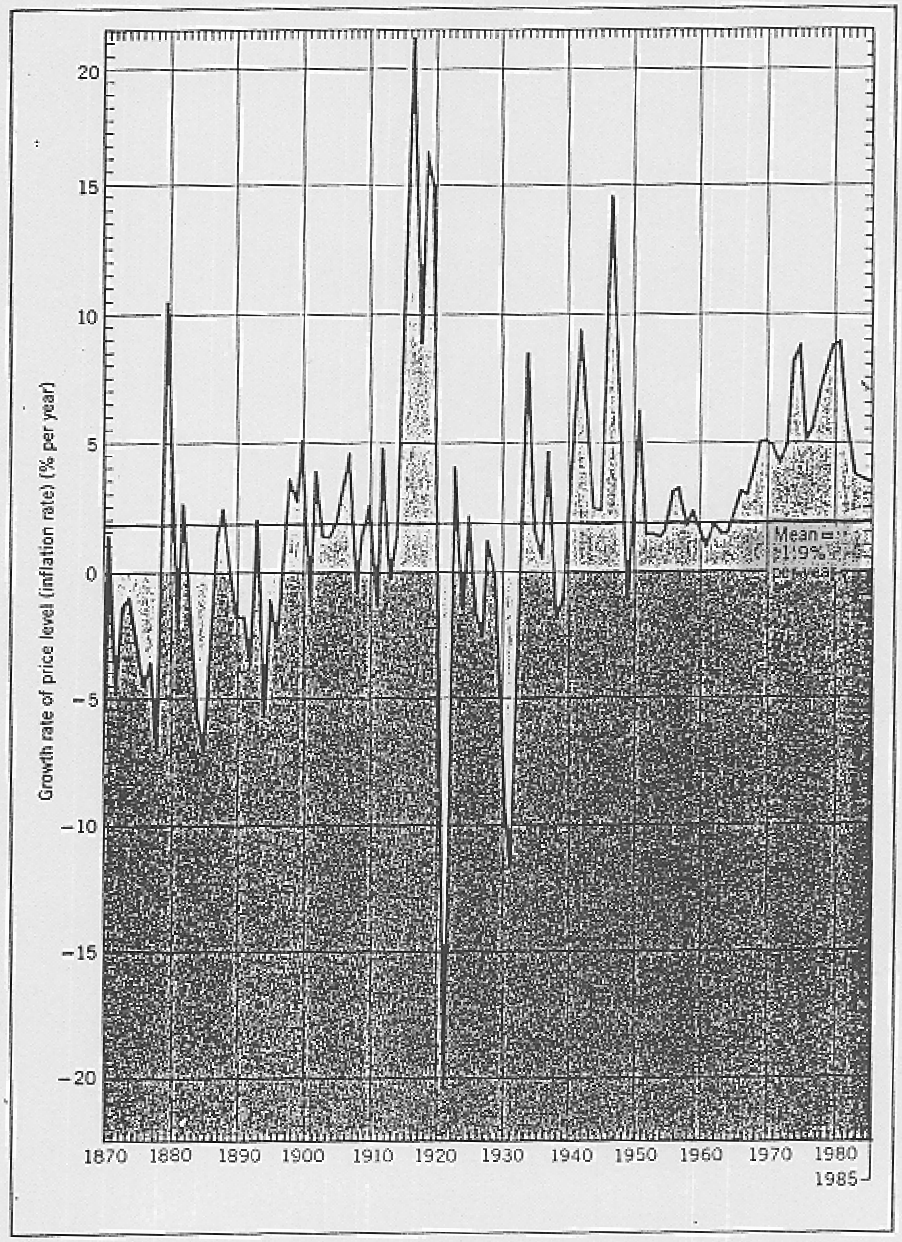


Figure 1.5 Inflation Rates in the United States, 1870–1985

previous series differenced
trend is gone
const. mean
regime switch 1950 – less volatile

Figures 3.12-3.21 EXAMPLE OF SEASONAL, TRADING-DAY, AND HOLIDAY ADJUSTMENT: RETAIL SALES OF MEN'S AND BOYS' CLOTHING

Figure 3.12 LOG RETAIL SALES (Z_t^*)

mensuelle, pic à la fin de l'année. Noel, janvier-bas après pic à les Pâques,

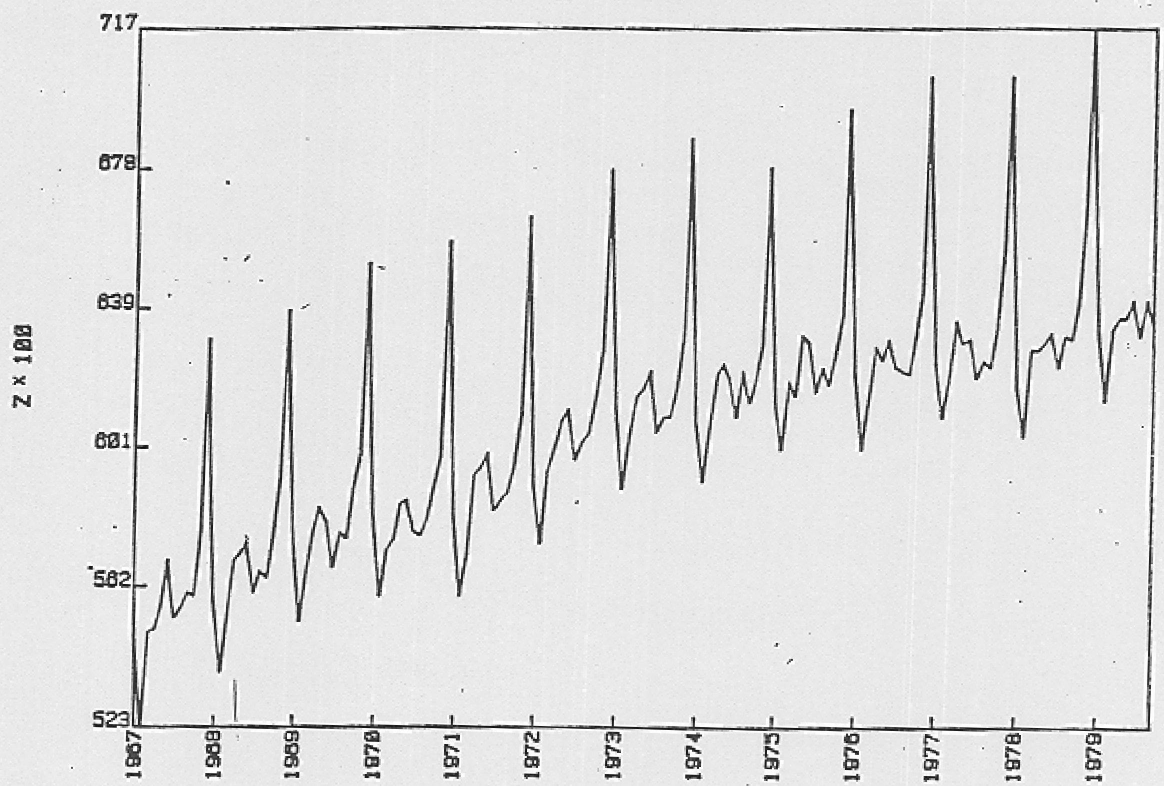


Figure 3.13 SACF OF Z_t^*

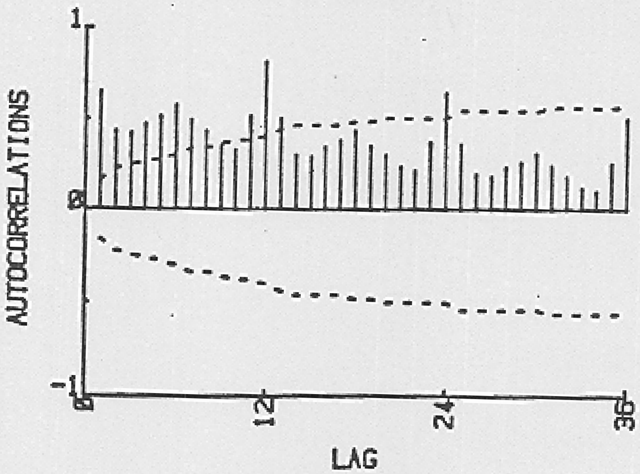
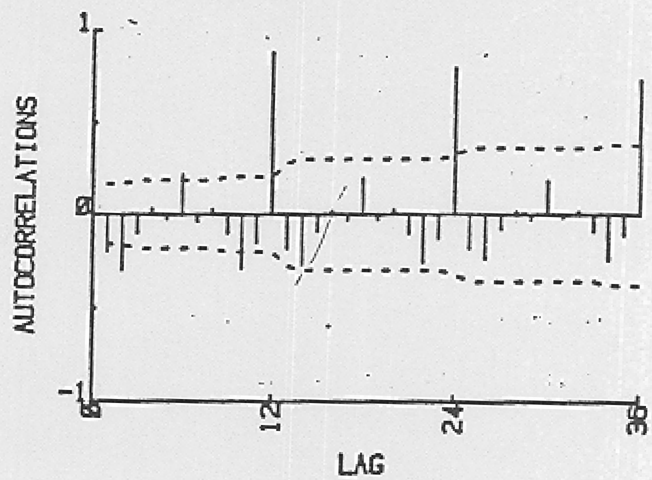
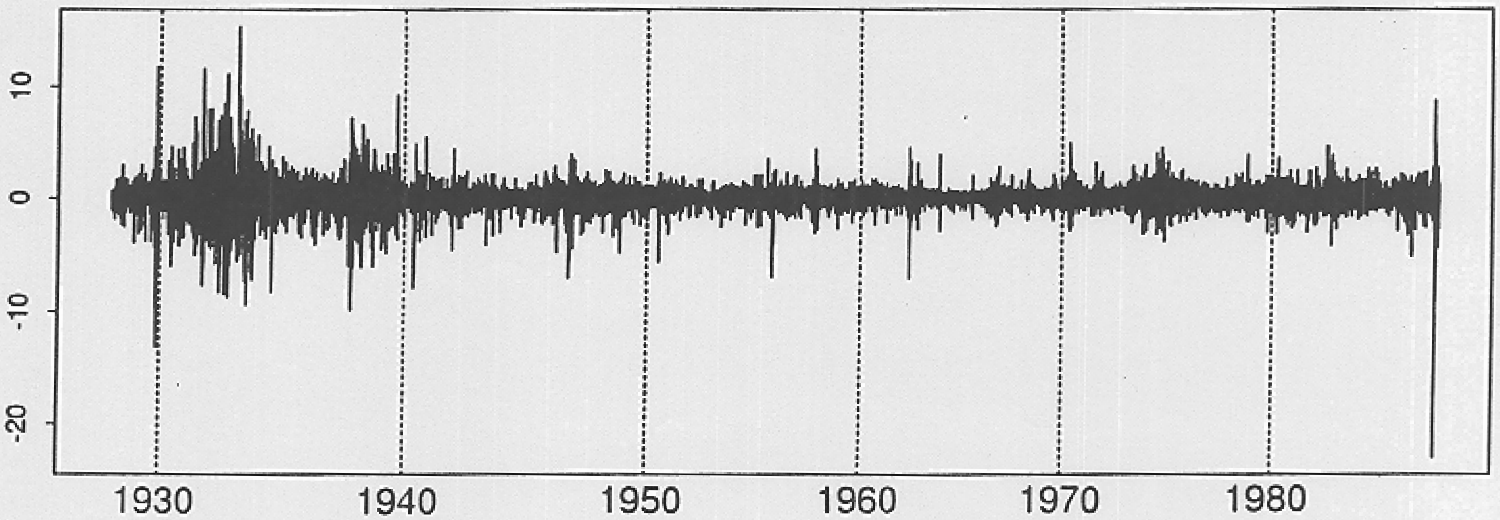


Figure 3.14 SACF OF $(1 - B)Z_t^*$



Unadjusted Price Movements, 100 ($\log P_t - \log P_{t-1}$)



const mean
time varying variance

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- MODELING TIME SERIES

OBJECTIVE: DESCRIBE THE BEHAVIOUR OF ONE OR SEVERAL T-S.

STAGES:
- MODEL SELECTION
- ESTIMATION
- TESTING

- IMPORTANT TOPICS

1. FORECASTING

GIVEN SOME OBS $X_1, \dots X_T$ WE WANT TO FIND $\hat{X}_{T+h}$

- POINT FORECAST $\hat{X}_T(h)$
- INTERVAL $[\hat{X}^1_T(h), \hat{X}^2_T(h)]$

EXTRAPOLATION $h > 0$
INTRAPOLATION $h < 0$
(filling in missing values)

2. DECOMPOSITION

1° ESTIMATE THE TREND
2° REMOVE THE TREND
3° ESTIMATE THE SEASONAL COMPONENT
4° REMOVE THE SEASONALITY

ex: ASSUME THAT X_t CAN BE WRITTEN AS:

$$X_t = Z_t + S_t + u_t$$

Where: Z_t: TREND, S_t: SEASONAL, U_t: RANDOM. Hence:

1° ESTIMATE Z_t

2° $X_t - Z_t$

3° S_t

4° $X_t - S_t$

3. DETECT AND MODEL THE STRUCTURAL CHANGES (REGIME SWITCHES)

4. STUDY THE DYNAMIC RELATIONSHIP BTW SOME VARIABLES; CAUSALITY

5. IDENTIFY THE SHORT TERM AND LONG TERM RELATIONS: COINTEGRATION

- A GENERAL OVERVIEW OF MODELS

→ DETERMINISTIC MODELS (NO RANDOMNESS)

- A DETERMINISTIC F. OF TIME: $X_t = f(t)$
- A RECURRENCE: $X_t = f(t, X_{t-1}, X_{t-2}, \dots)$

YIELDS A PERFECT FORECAST IF PAST VALUES ARE KNOWN.

→ STOCHASTIC MODELS

1. TREND MODEL

$$X_t = f(t, u_t)$$

t: time

u_t: random noise, independent

ex: $X_t = g(t) + u_t$

$X_t = g(t) \cdot u_t$

g: deterministic (or stochastic)

• examples

$$f(t) = A_0 + \sum_{j=1}^{q} \left[A_j \cos(\omega_j t) + B_j \sin(\omega_j t) \right]$$

periodic function

$f(t) = \beta_0 + \beta_1 t$ linear

$f(t) = \beta_0 + \beta_1 t + \beta_2 t^2 + \dots + \beta_K t^K$ polynomial

$f(t) = \beta_0 + \beta_1 r^t$ exponential : depending on r converge or explode

IN GENERAL 2 METHODS TO ESTIMATE AND REMOVE TREND :

a) GLOBAL METHODS : WHERE ALL OBS ARE GIVEN EQUAL WEIGTHS

b) LOCAL ADJUSTEMENTS : OBS CLOSE IN TIME ARE GIVEN MORE WEIGHT

MOVING AVERAGES

MA:

$$X_t^* = \theta_{-m_1} X_{t-m_1} + \theta_{-m_1+1} X_{t-m_1} + \cdots + \theta_{m_2} X_{t+m_2}$$

$$X_t^* = \sum_{i=-m_1}^{m_2} \theta_i X_{t+i}$$

EXPONENTIAL SMOOTHING

$$X_t^* = \beta X_{t-1}^* + (1-\beta) X_t$$

- IN ECONOMICS WE USE PEARSON'S DECOMPOSITION

$$X_t = Z_t + C_t + S_t + u_t$$

WHERE C_t IS THE BUSINESS CYCLE

2. FILTER MODELS (GENERALIZED MOVING AVERAGES)

$$X_t = f(\ldots, u_{t-1}, u_t, u_{t+1}, \ldots)$$

where u_t independent or non-correlated

IMPORTANT CASE: MOVING AVERAGE OF ORDER q

$$X_t = \bar{\mu} + u_t - \sum_{j=1}^{q} \theta_j u_{t-j}$$

3. AUTO PROJECTION MODELS

$$X_t = f(X_{t-1}, X_{t-2}, \dots, u_t)$$

u_t: noise as in $MA(q)$

IMPORTANT CASE: AUTOREGRESSIVE OF ORDER p:

$$X_t = \bar{\mu} + \sum_{j=1}^{p} \varphi_j X_{t-j} + u_t$$

4. EXPLANATORY MODELS

$$X_t = f(Z_t^*, u_t)$$

Z_t^* CONSISTS OF EXPLANATORY VAR (EXOGEN.) AND POSSIBLY PAST VALUES OF X_t

STOCHASTIC PROCESSES

- A PROBABILITY SPACE $(\Omega, \mathcal{A}, P)$ WHERE:

Ω SAMPLE SPACE

$\mathcal{A}$ σ ALGEBRA OF SUBSETS OF Ω, i.e:

(i) $\Omega \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) for any sequence $\{A_1, A_2 \dots\} \subseteq \mathcal{A}$

$$\bigcup_{j=1}^{\infty} A_j \in \mathcal{A}$$

$P: \mathcal{A} \to [0, 1]$ A FUNCTION THAT ASSIGNS TO $A \in \mathcal{A}$ A VALUE $P(A) \in [0,1]$ SUCH THAT $P(\Omega) = 1$ AND IF $\{A_j\}_{j=1}^{\infty}$ IS A SEQUENCE OF DISJOINT EVENTS

$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

- RANDOM VARIABLE (r.v.) IS A FUNCTION

$X : \Omega \to \mathbb{R}$ SUCH THAT

$$X^{-1}\big((-\infty, x)\big) = \{\omega \in \Omega : X(\omega) \le x\} \in \mathcal{A} \quad \forall x \in \mathbb{R}$$

MEASURABLE FUNCTION

THE DISTRIBUTION

$$F_X(x) = P[X^{-1}((-\infty, x])]$$
$$= P[w: X(w) \le x]$$

STOCHASTIC PROCESS

$\{X(t)\}$ or $\{X_t\}$ IS A FAMILY OF RANDOM VARIABLES INDEXED BY t, WHERE $t \in T$. IF $T = \mathbb{R}$ CONTINUOUS TIME

$T = \mathbb{Z}$ DISCRETE TIME PROCESS

THE PROBABILITY LAW OF $\{X_t, t \in T\}$ CAN BE DESCRIBED BY SPECIFYING FOR EVERY SUBSET $\{t_1, t_2, \dots t_n\} \subseteq T$ WHERE $n \ge 1$, A JOINT DISTRIBUTION OF $(X_{t_1}, \dots X_{t_n})$

$$F_{X_{t_1} \dots X_{t_n}}(x_1 \dots x_n) = P[X_{t_1} \le x_1, \dots X_{t_n} \le x_n]$$

(KOLMOGOROV)

L_r SPACES

LET r A REAL NUMBER. L_r IS THE SET OF ALL REAL r.v. X on $(\Omega, \mathcal{F}, P)$ SUCH THAT $E[|X|^r] < \infty$

i.e. r^{th} moment exists

IMPORTANT: L_2 IS THE SET OF r.v. on $(\Omega, \mathcal{A}, P)$ WITH FINITE SECOND MOMENTS

$$L_1 \subset L_2 \subset L_3$$

A STOCHASTIC PROCESS $\{X_t, t \in T\}$ IS IN L_2 IFF $X_t \in L_2$ $\forall t \in T$, i.e.

$$E[\,|X_t|^2\,] < \infty \qquad \forall t \in T$$

- STATIONARITY

$\{X_t\}$ IS STRICTLY STATIONARY IF FOR ANY ADMISSIBLE $t_1, t_2 \dots t_n$ AND ANY POSITIVE INTEGER n THE JOINT PROBABILITY DISTRIBUTION OF

$$\{X_{t_1}, X_{t_2}, \dots X_{t_n}\}$$

IS IDENTICAL TO THE JOINT DISTRIBUTION OF

$$\{X_{t_1+K}, X_{t_2+K}, \dots X_{t_n+K}\}$$

$K \in \mathbb{Z}$; i.e.

$$F_{X_{t_1}, \dots X_{t_n}}(x_1 \dots x_n) = F_{X_{t_1+K}, \dots X_{t_n+K}}(x_1, \dots x_n)$$

→ INVARIANCE OF THE PROBABILISTIC STRUCTURE UNDER A SHIFT OF THE TIME ORIGIN

ASSUME $E(X_t^2) < \infty \ \forall t \in T$. IF $\{X_t\}$ is SS, then

- $E(X_s) = E(X_t) \quad \forall s,t \in T$
- $E(X_s X_t) = E(X_{s+k} X_{t+k}) \quad \forall s,t \in T, \ \forall k \geq 0$

AND SINCE

$$COV(X_s, X_t) = E(X_s X_t) - E(X_s)E(X_t)$$

WE ALSO HAVE

$$COV(X_s X_t) = COV(X_{s+k} X_{t+k}) \quad \forall s,t \in T, \ \forall k \geq 0$$

MEANS THAT THE MEAN OF X_t IS CONSTANT AND THE COVARIANCE BTW ANY 2 VARIABLES DEPEND ONLY ON THE DISTANCE BTW THESE VARIABLES AND NOT ON THEIR TIMING (POSITION IN TIME)

SECOND ORDER STATIONARITY (WEAK, WIDE-SENSE, COVARIANCE STATIONARY

$\{X_t\}$ IS SECOND ORDER STATIONARY IF

(1) $E(X_t^2) < \infty, \ \forall t \in T$

(2) $E(X_s) = E(X_t), \ \forall t,s \in T$

(3) $COV(X_s, X_t) = COV(X_{s+k}, X_{t+k}) \quad \forall s,t \in T, \ \forall k \geq 0$

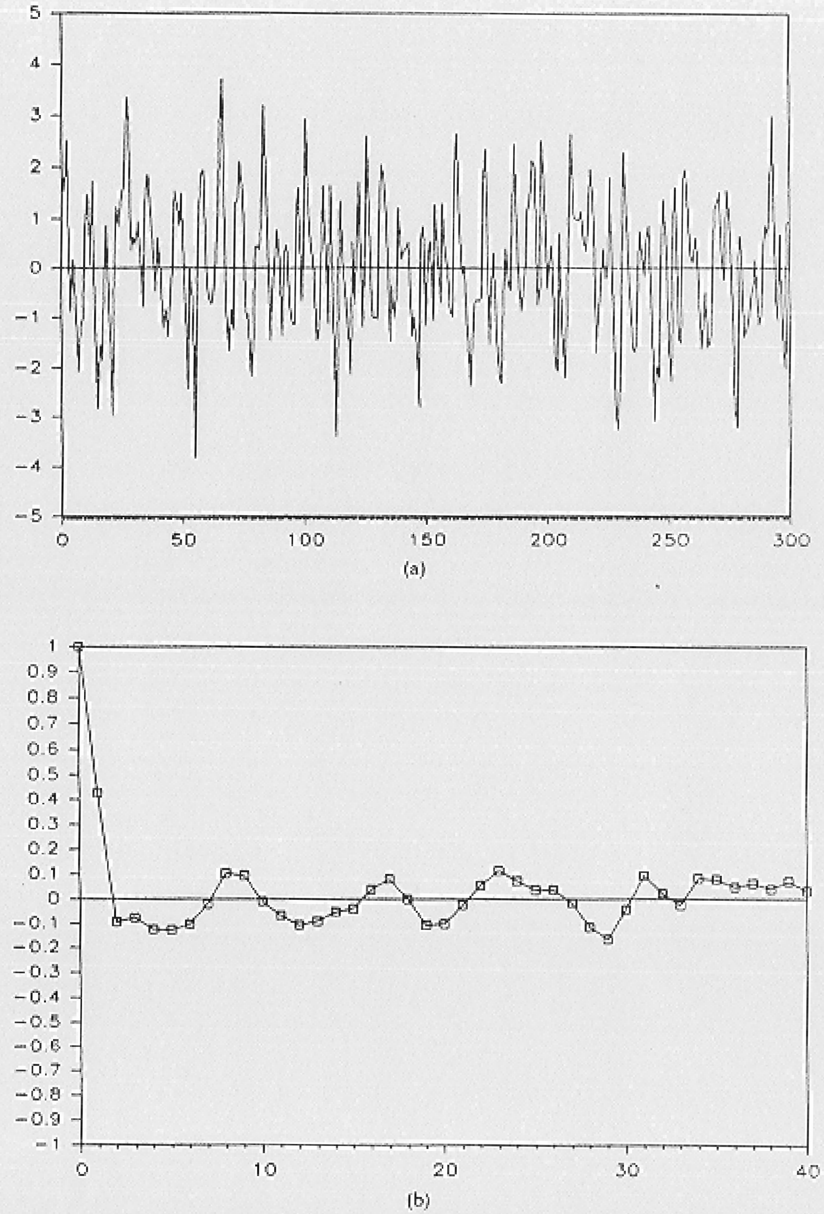


Figure 1.18. (a) 300 observations of the series $X_t = Z_t + .95Z_{t-1}$, Example 1.5.2. (b) The sample autocorrelation function $\hat{\rho}(h)$, $0 \le h \le 40$.

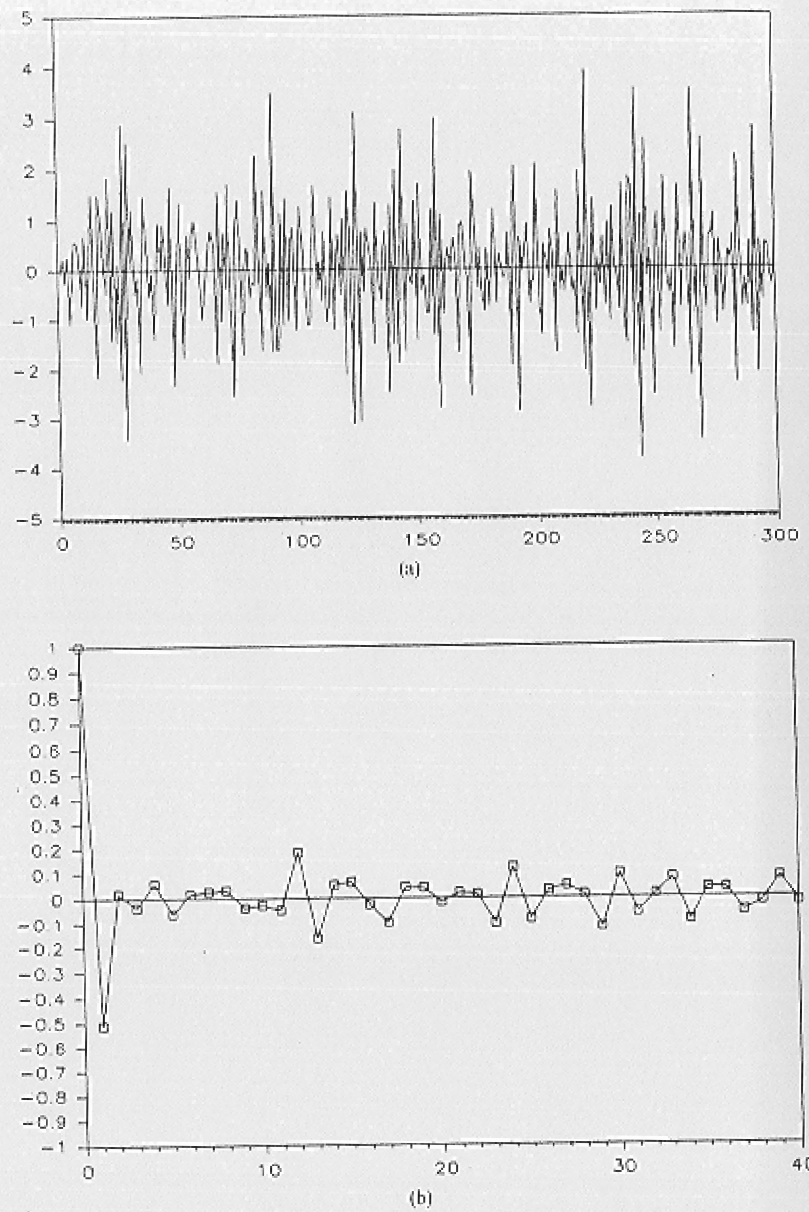
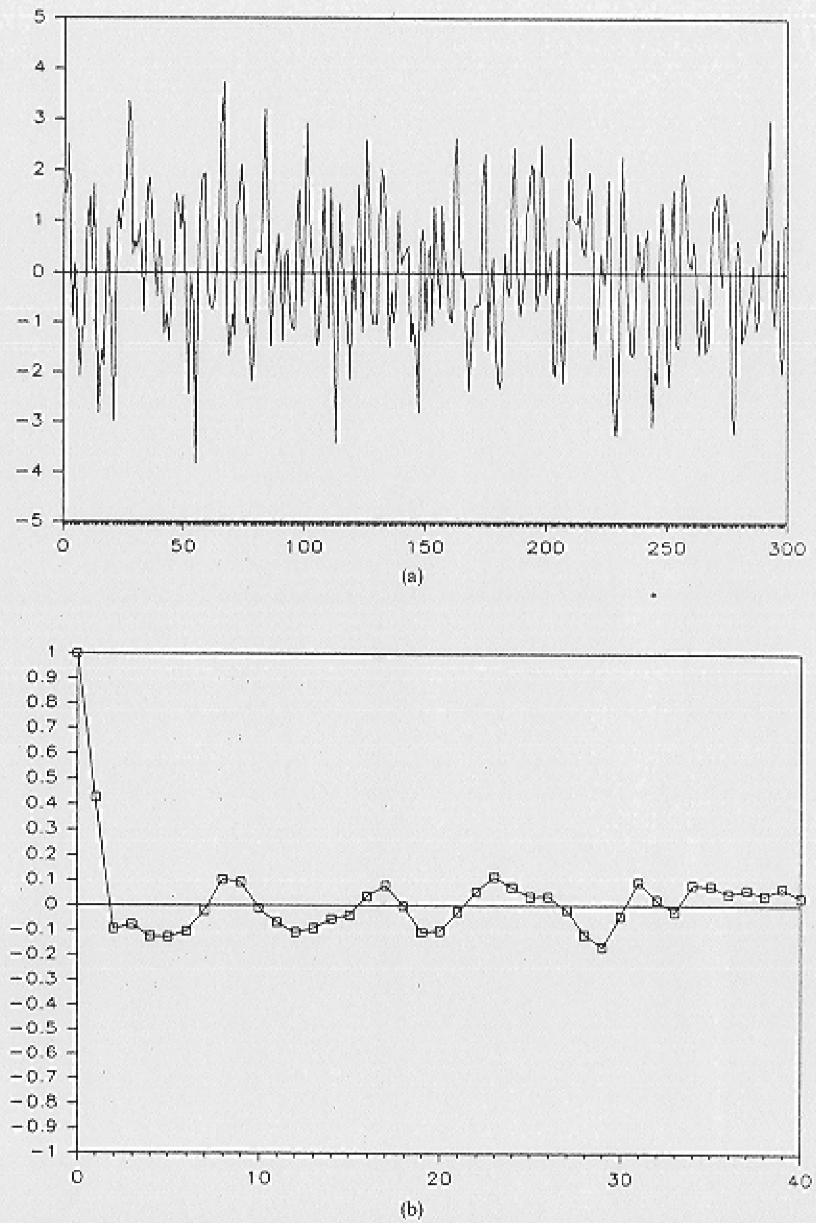


Figure 1.19. (a) 300 observations of the series $X_t = Z_t - .95Z_{t-1}$, Example 1.5.3. (b) The sample autocorrelation function $\hat{\rho}(h)$, $0 \le h \le 40$.



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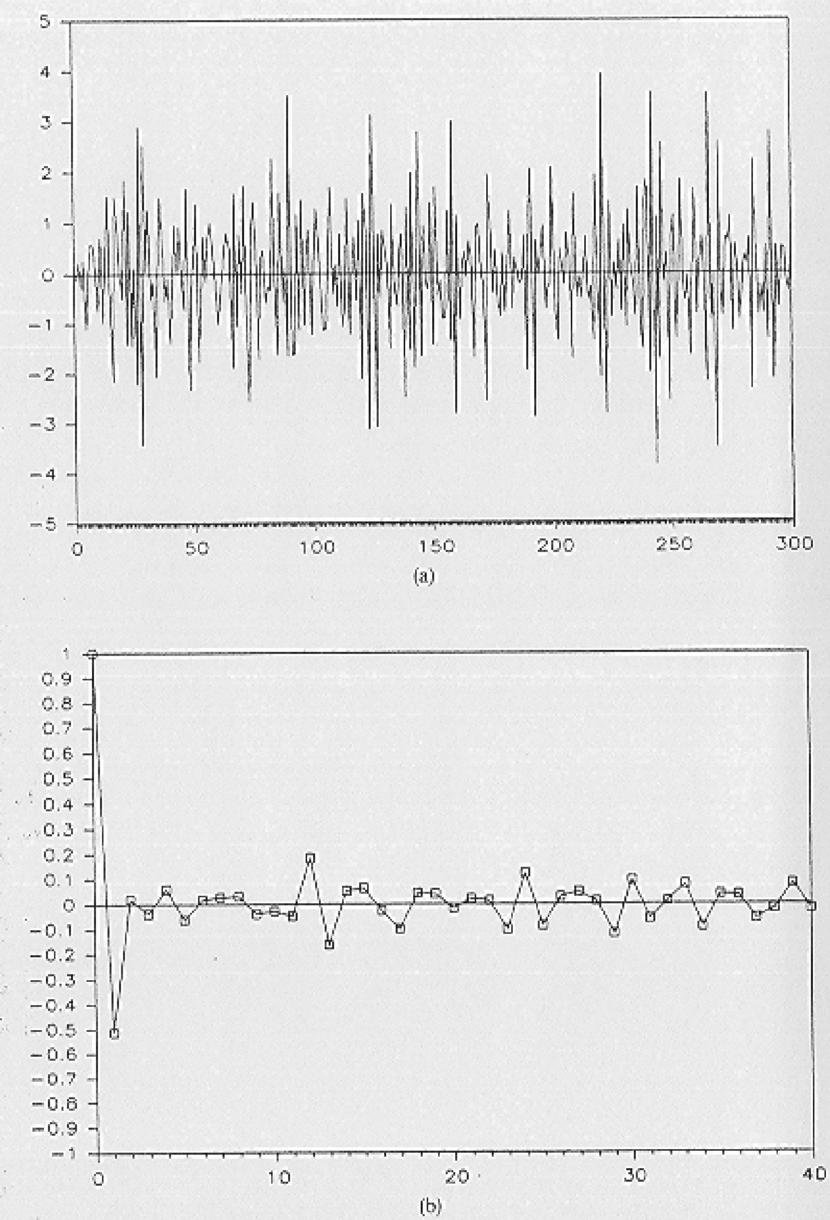

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- $E(X_s) = E(X_t) \quad \forall s,t \in T$
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AND SINCE

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$$COV(X_s X_t) = COV(X_{s+k} X_{t+k}) \quad \forall s,t \in T, \ \forall k \geq 0$$

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(3) $COV(X_s, X_t) = COV(X_{s+k}, X_{t+k}) \quad \forall s,t \in T, \ \forall k \geq 0$

i.e.

$E(X_t) = \mu$ CONSTANT, INDEPEND OF TIME t

$E(X_t^2) = \mu_2$ SAME

hence

$\operatorname{var}(X_t) = \mu_2 - \mu^2 = \sigma^2$ CONSTANT, INDEP OF t

→ MEAN, VARIANCE CONSTANT

→ COVARIANCE DEPENDS ON $(t-s)$ ONLY

RELATION BTW STRICT STATIONARITY AND 2nd ORDER STATION.

IN GENERAL: STRICT STATIONARITY $\not\Rightarrow$ STATIONARITY

STATIONARITY $\not\Rightarrow$ STRICT STATIONARITY

HOWEVER:

(a) IF MOMENTS OF X_t EXIST UPTO ORDER 2:

STRICT STATIONARITY $\Rightarrow$ STATIONARITY

(b) IF X_t IS A STATIONARY GAUSSIAN PROCESS I.E. $X_{t_1}, X_{t_2}, \dots, X_{t_n}$ IS MULTIVARIATE NORMAL, THAN X_t IS STRICTLY STATIONARY

GAUSSIAN DISTRIBUTION COMPLETEY CHARACTERIZED BY THE FIRST TWO MOMENTS

ESTIMATION AND ELIMINATION OF TREND TO ACHIVEVE STATIO

$$y_t = Z_t + S_t + u_t;$$

u_t random, noise, weakly stat.

ASSUME S_t HAS BEEN CORRECTLY REMOVED

Let $Z_t = f(t)$, i.e.

$$Z_t = a_0 + a_1 t + a_2 t^2$$

① ESTIMATE BY OLS

$$\hat{y}_t = \hat{a}_0 + \hat{a}_1 t + \hat{a}_2 t^2 + \hat{u}_t$$

→ THE RESIDUAL $\hat{u}_t$ IS STATIONARY

② METHOD OF DIFFERENCING

Δ: DIFFERENCE OPERATOR

$$\Delta y_{t+h} = y_{t+h} - y_t$$

$$\Delta y_t = y_t - y_{t-1}$$

$$\Delta y_{t+1} = y_{t+1} - y_t$$

$$\Delta^2 y_t = \Delta(\Delta y_t) = \Delta(y_t - y_{t-1}) = (y_t - y_{t-1}) - (y_{t-1} - y_{t-2})$$

$$= y_t - 2y_{t-1} - y_{t-2}$$

L: LAG OPERATOR

$$L^h y_t = y_{t-h}$$

$$Lc = c \qquad c = \text{CONSTANT}$$

$$L y_t = y_{t-1}$$

FOR $h = -2, -1, 0, 1, 2$. IF $h < 0$ YOU SHIFT FORWARD IN TIME !

HENCE THE METHOD OF DIFFERENCING CONSISTS ON APPLYING AN ORDER d DIFFERENCE OPERATION ON Y_t, i.e.:

$$\Delta^d Y_t = (1-L)^d Y_t = (1-\_)^d Z_t + (1-L)^d u_t$$

CHOSE SUCH d THAT $(1-L)^d Z_t$ IS A CONSTANT AND $(1-L)^d u_t$ IS A STATIONARY PROCESS. USUALLY $d=1$ OR $d=2$

CHOICE BTW ① OR ② : UNIT ROOT !

AUTOCOVARIANCE FUNCTION

IF $\{X_t\}$ IS 2nd ORDER STATIONARY, THERE EXISTS A FUNCTION

$$\gamma : \mathbb{Z} \to \mathbb{R}$$

SUCH THAT

$$\operatorname{cov}(X_s, X_t) = \gamma(t-s) \qquad \forall\, s,t \in T$$

WE CALL $\gamma(K)$ A COVARIANCE FUNCTION WHERE $K = t - S$

$\gamma(0)$ IS THE VARIANCE

FOR A SERIES STATIONARY WITH MEAN μ AND VARIANCE σ^2

$$\gamma(K) = E\left[(X_t - \mu)(X_{t+K} - \mu) \right]$$

PROPERTIES $\Rightarrow$

AN EVEN FUNCTION

Let $r \in T$ and $s \leq t$

if $s \geq r$

$$\mathrm{cov}(X_r, X_{r+t-s}) = \mathrm{cov}(X_{r+s-r}, X_{r+t-s+s-r}) = \mathrm{cov}(X_s, X_t)$$

if $s < r$

$$\mathrm{cov}(X_s, X_t) = \mathrm{cov}(X_{s+r-s}, X_{t+r-s}) = \mathrm{cov}(X_r, X_{r+t-s})$$

let $s > t$

$$\mathrm{cov}(X_s, X_t) = \mathrm{cov}(X_t, X_s) = \mathrm{cov}(X_r, X_{r+s-t})$$

hence:

$$\mathrm{cov}(X_s, X_t) = \mathrm{cov}(X_r, X_{r+|t-s|}) = \gamma(t-s)$$

FOR ANY $\gamma(K)$: $\gamma : \mathbb{Z} \to \mathbb{R}$

(1) $\gamma(0) = \mathrm{var}(X_t) \geq 0 \qquad \forall\, t \in T$

(2) $\gamma(K) = \gamma(-K) \qquad \forall\, K \in \mathbb{Z}$

(3) $|\gamma(K)| \leq \gamma(0) \qquad \forall\, K \in \mathbb{Z}$

(4) $\gamma(K)$ IS POSITIVE DEFINITE :

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \gamma(t_i - t_j) \geq 0$$

for all positive integers N and vectors $a=(a_1 \dots a_N)' \in \mathbb{R}^N$ and $(t_1 \dots t_N)' \in T^N$

proof:

$$0 \leq \operatorname{var}\left(\sum_{i=1}^{N} a_i X_{t_i}\right) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \operatorname{cov}(X_{t_i}, X_{t_j}) =$$

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \gamma(t_i - t_j) = a' \Gamma_N a$$

$\Rightarrow$ (5)

ANY N×N MATRIX

$$\begin{bmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{N-1} \\ \gamma_1 & \gamma_0 & \gamma_1 & \cdots & \gamma_{N-2} \\ \vdots & & \ddots & & \\ \gamma_{N-1} & \gamma_{N-2} & \gamma_{N-3} & & \gamma_0 \end{bmatrix} = \Gamma_N = \left[\gamma(j-i)\right]_{i,j=1,\dots N}$$

IS POSITIVE DEFINITE

<u>TREOREM</u>: A REAL VALUED EVEN FUNCTION DEFINED ON THE SET $\mathbb{Z}$ OF ALL INTEGERS IS NON-NEGATIVE DEFINITE IF AND ONLY IF IT IS THE AUTOCOVARIANCE FUNCTION OF A STATIONARY TIME SERIES

- IF $VAR(X_t) = \gamma(0) = 0$ the process is deterministic
- IF Γ_N SINGULAR

AUTOCORRELATION FUNCTION

$$\rho(K) = \frac{\gamma(K)}{\gamma(0)}$$

IF X_t IS STATIONARY, THERE EXISTS $\rho: \mathbb{Z} \to [-1,1]$ SUCH THAT

$\rho(t-s) = \operatorname{corr}(X_s, X_t) = \gamma(t-s)/\gamma(0) \quad \forall\, s,t \in T$

PROPERTIES

(1) $\rho(0) = 1$

(2) $\rho(K) = \rho(-K) \qquad \forall K \in \mathbb{Z}$

(3) $|\rho(K)| \leq 1 \qquad \forall K \in \mathbb{Z}$

(4) $\rho(K)$ IS POSITIVE SEMI-DEFINITE, i.e.

$$\sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \, \rho(t_i - t_j) \geq 0$$

for any N, $a = (a_1 \dots a_N)' \in \mathbb{R}^N$ and $\tau = (t_1, \dots, t_N)' \in T^N$

(5) ANY NxN MATRIX

$$R_N = \frac{1}{\gamma_0} \Gamma_N = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \dots & \rho_{N-1} \\ \rho_1 & 1 & \rho_1 & \dots & \rho_{N-2} \\ \vdots & & \ddots & & \\ \rho_{N-1} & \rho_{N-2} & \rho_{N-3} & \dots & 1 \end{bmatrix}$$

IS POSITIVE SEMI DEFINITE ($\gamma_0 = \operatorname{var}(X_t)$, $\rho_K = \rho(K)$)

STATIONARITY UP TO ORDER m

Let m be a non-negative integer. We say $\{X_t\}$ IS STATIONARY OF ORDER M IFF

(i) $E(|X_t|^m) < \infty \quad \forall t \in T$

(2) $E\left[X_{t_1}^{m_1} X_{t_2}^{m_2} \dots X_{t_n}^{m_n}\right] = E\left[X_{t_1+K}^{m_1} X_{t_2+K}^{m_2} \dots X_{t_n+K}^{m_n}\right]$

FOR ANY $K \geq 0$, $\{t_1, \dots, t_n\} \in T^N$ AND NON-NEGATIVE INTEGERS $m_1, \dots, m_n$ SUCH THAT $m_1 + m_2 + \dots + m_n \leq m$

$\Rightarrow$ IF $m=1$, $E(|X_t|) < \infty$

$E(X_{t_1}) = E(X_{t_1+K})$

$\Rightarrow$ IF $m=2$, $E(|X_t|^2) < \infty \Rightarrow E(X_t^2) < \infty$

$E(X_{t_1}) = E(X_{t_1+K})$

$E(X_{t_1}^2) = E(X_{t_1+K})^2$

$E(X_{t_1} X_{t_2}) = E(X_{t_1+K} X_{t_2+K})$ STATIONARITY

ARMA (p,q)

DEF:

ARMA(p,q) IS DEFINED AS $\{X_t\}$ SUCH THAT

$$A(L)\, X_t = B(L)\, e_t$$

WHERE e_t IS i.i.d. $(0, \sigma_e^2)$

$A(L) = 1 + a_1 L + a_2 L^2 + \dots + a_p L^p$ p^{th} order polynomial in L

$B(L) = 1 + b_1 L + b_2 L^2 + \dots + b_q L^q$ q^{th} order

NOTE THAT IF $\{X_t\}$ IS ARMA(p,q) WITH MEAN μ, THAN $\{X_t - \mu\}$ IS ARMA(p,q) TOO.

FROM NOW ON WE ASSUME $\{X_t\}$ IS "DEMEANED"

CONVENTIONALLY WE WRITE:

$$X_t = \mu + \sum_{j=1}^{p} a_j X_{t-j} + e_t - \sum_{i=1}^{q} b_i e_{t-i}$$

$$X_t - \sum_{j=1}^{p} a_j L^j X_t = \mu + e_t - \sum_{i=1}^{q} b_i L^i e_t$$

$$X_t \left(1 - \sum_{j=1}^{p} a_j L^j\right) = \mu + e_t\left(1 - \sum_{i=1}^{q} b_i L^i\right)$$

$$E(X_t) = E\left(\frac{\mu}{1 - \sum_{j=1}^{p} a_j}\right)$$

SINCE $E(e_t) = 0$. ALSO $E(X_t) < 0$ IF $1 - \sum_{j=1}^{p} a_j \neq 0$

WE WILL NOW

- FIND CONDITIONS ON $A(L)$ AND $B(L)$ WHICH ENSURE THAT $\{x_t\}$ IS STATIONARY
- HIGHLIGHT THE PROPERTIES OF THE AUTOCOVAR F IN SOME SPECIAL CASES

A. WHITE NOISE PROCESS

$$A(L) = B(L) = 1$$

DEF $\{x_t\}$ IS A W.N. IF IT CONSISTS OF A SEQUENCE OF UNCORRELATED R.V. IN L_2

- SOMETIMES INDEPENDENCE IS REQUIRED : STRONG W.N.
- EVENTUALLY NORMALITY IS IMPOSED: GAUSSIAN W.N.

FOR A W.N. TO BE STATIONARY WE REQUIRE

$$E(x_t) = \mu$$
$$E[(x_t - \mu)^2] = \sigma^2$$

BOTH FINITE

CONVENTIONALLY $\mu = 0$

$$\gamma(k) = \begin{cases} 0 & \text{if } k \neq 0 \\ \sigma^2 & \text{if } k = 0 \end{cases}, \quad \rho(k) = \begin{cases} 0 & \text{if } k \neq 0 \\ 1 & k = 0 \end{cases}$$

AND IDENTICALLY DISTRIBUTED X_t

B. FIRST ORDER AUTOREGRESSIVE MODEL : AR(1)

$B(L) = 1$

$A(L) = 1 - aL$

$$x_t = a x_{t-1} + e_t$$

$$e_t \sim i.i.d. \left(0, \sigma_e^2\right)$$

SPECIAL CASE OF A MARKOV PROCESS

$$P(x_t \mid x_{t-s} \,;\, s \geq 1) = P(x_t \mid x_{t-1})$$

THE STATIONARITY OF x_t DEPENDS ON a

– and x_0, the initial condition.

ASSUME x_0 INDEPENDENT OF e_t $\forall t$.

$$x_t = a x_{t-1} + e_t$$

$$= a\,(a x_{t-1} + e_{t-1}) + e_t$$

$$\vdots$$

$$x_t = a^t x_0 + \sum_{i=0}^{t-1} a^i e_{t-i}$$

Hence:

(a) $E(x_t) = a^t E(x_0)$

(b) $Var(x_t) = a^{2t}\, var(x_0) + \sigma_e^2 \sum_{i=0}^{t-1} (a^2)^i +$

$$+\ 2\, cov\left(a^t x_0, \sum_{i=0}^{t-1} a^i e_{t-i}\right.$$

$$= a^{2t} \operatorname{var}(x_0) + \sigma_e^2 \left(\frac{1-(a^2)^t}{1-a^2}\right), \quad \text{if } |a| \neq 1$$

$$= \operatorname{var}(x_0) + \sigma_e^2 t, \quad \text{if } |a| = 1$$

using: for $S_t = 1 + a^2 + a^3 + \cdots + a^{t-1}$

$$= \frac{1-a^t}{1-a} \quad \text{if } a \neq 1$$

$$= t \quad \text{if } a = 1$$

(c) $$E(x_t X_{t+\tau}) = E\left\{\left[a^t X_0 + \sum_{i=0}^{t-1} a^i e_{t-i}\right]\left[a^{t+\tau} X_0 + \sum_{i=0}^{t+\tau-1} a^i e_{t-i+\tau}\right]\right.$$

$$= a^{2t+\tau} E(x_0^2) + \sigma_e^2 \left[a^\tau + a^{\tau+2} + \cdots + a^{\tau+2(t-1)}\right]$$

since $E(e_t e_s) = 0$ for $t \neq s$
$= \sigma_e^2 \quad t = s$

Check for ex: $\left(\sum_{i=0}^{3} a^i e_{t-i}\right)\left(\sum_{i=0}^{3+2} a^i e_{t-i+2}\right)$

$$= a^{2t+\tau} E(x_0^2) + \sigma_e^2 a^\tau \left(\frac{1-a^{2t}}{1-a^2}\right), \quad \text{if } |a| \neq 1$$

$$= a^{2t+\tau} E(x_0^2) + \sigma_e^2 t, \quad \text{if } |a| = 1$$

CLAIM:

$\{x_t\}$ IS STATIONARY IF $x_0 \sim \left(0, \frac{\sigma_e^2}{1-a^2}\right)$ AND DISTRIBUTED INDEPENDENTLY OF e_t, AND IF $|a|<1$

PROOF:

(1) $E(x_t)=0$, INDEPENDENT OF t

(2) $var(x_t) = a^{2t}\frac{\sigma_e^2}{1-a^2} + \sigma_e^2 \frac{1-a^{2t}}{1-a^2} = \frac{\sigma_e^2}{1-a^2}$, INDEP OF t

(3) $$E(x_t\, x_{t+\tau}) = a^{2t+2}\frac{\sigma_e^2}{1-a^2} + \sigma_e^2 a^{\tau}\frac{(1-a^{2t})}{1-a^2}$$

$$= \sigma_e^2 \frac{a^{|\tau|}}{1-a^2}$$, INDEP OF t

FOR ALL OTHER CASES x_t IS NOT STATIONARY. **HOWEVER IF**

$|a| < 1$

THE AR(1) IS ASYMPTOTICALLY STATIONARY FOR ALMOST ANY SPECIFICATION OF x_0

TAKE LIMITS AS $t \to \infty$

(1) $\lim_{t\to\infty}\left(a^t E(x_0)\right) = 0$

(2) $\lim_{t\to\infty}\left(a^{2t} var(x_0) + \sigma_e^2 \sum_{i=0}^{\infty}(a^2)^i\right) = \sigma_e^2\frac{1}{1-a^2} = \frac{\sigma_e^2}{1-a^2}$

(3) $$\lim_{t \to \infty} (\mathrm{cov}) = \sigma_e^2 \frac{a^{|k|}}{1-a^2}$$

AR(1) IS STATIONARY IF $|a| < 1$

IN THIS CASE
$$\gamma(K) = \sigma_e^2 \frac{a^{|K|}}{(1-a^2)}$$

$$\rho(K) = a^{|K|}$$

FOR ANY AR(p) MODEL WE DERIVE A GENERAL RESULT USING:

DIFFERENCE EQUATIONS

Consider a difference equation:

(1) $$Y_t = a + \lambda Y_{t-1} + b X_t$$

FORCING VAR
FOR US: A W.N.

ASSUME $\lambda \neq 1$

(2) $$(1-\lambda L) Y_t = a + b X_t$$

$$Y_t = \underbrace{\frac{a}{1-\lambda L} + \frac{b}{1-\lambda L} X_t}_{\text{SOLUTION TO (1)}} + \underbrace{c\lambda^t}_{\text{SOLUTION TO}}$$

as we can see

$$(1-\lambda L)\, c\lambda^t = c\lambda^t - c\lambda\cdot\lambda^{t-1} = 0$$

(L shifts power too!)

to eliminate the arbitrary constant c we use the initial condition

$$y_0 = \frac{a}{1-\lambda} + b\sum_{i=0}^{\infty} \lambda_i X_{0-i} + c\lambda^0$$

WE PROCEDE AS FOLLOWS

$$Y_t = a\sum_{i=0}^{t-1}\lambda^i + a\sum_{i=t}^{\infty}\lambda^i + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + b\sum_{i=t}^{\infty}\lambda^i X_{t-i} + c\lambda^t$$

$$= \frac{a(1-\lambda^t)}{1-\lambda} + \frac{a\lambda^t}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + b\lambda^t\sum_{i=0}^{\infty}\lambda^i X_{0-i} + c\lambda$$

given that $\sum_{i=0}^{T} a^i = \frac{1-(a)^{T+1}}{1-a}$ if $|a| \neq 1$

$$\sum_{i=t}^{\infty}\lambda^i = \lambda^t\sum_{i=0}^{\infty}\lambda^i = \lambda^t\cdot\frac{1}{1-\lambda}$$

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t\left\{\frac{a}{1-\lambda} + b\sum_{i=0}^{\infty}\lambda^i X_{0-i} + c\lambda^0\right\}$$

$$Y_t = \frac{a(1-\lambda^t)}{1-\lambda} + b\sum_{i=0}^{t-1}\lambda^i X_{t-i} + \lambda^t Y_0$$

Assume $X_t = 0$ for all $t \geq 0$ for simplicity

$$Y_t = \frac{a}{1-\lambda} + \lambda^t \left(Y_0 - \frac{a}{1-\lambda} \right)$$

IT IS A SOLUTION TO $Y_t = a + \lambda Y_{t-1}$ S.T. Y_0 AS INITIAL CONDITION

• IF $Y_0 = \dfrac{a}{1-\lambda}$ $\Rightarrow$ $Y_t = Y_0$ for all $t \geq 0$, SO THAT

$\dfrac{a}{1-\lambda}$ IS A STATIONARY PT. LONG RUN EQUILIBRIUM

• IF Y_0 IS DIFFERENT

IF $|\lambda| < 1$, THAN:

$$\lim_{t \to \infty} Y_t = \frac{a}{1-\lambda}$$

SYSTEM IS STABLE, TENDS TO APPROACH THE STATIONARY PT IN THE LONG RUN.

the root $|\lambda| < 1$ IS CALLED STABLE ROOT

$|\lambda| > 1$ UNSTABLE

• IF ADDITIONALY $a = 0$

$$Y_t = \lambda^t Y_0$$

0 is THE STATIONARY PT.

IF $|\lambda| > 1$ SYSTEM DIVERGE FROM THE STATIONARY PT.

$\lambda > 1$ $\Rightarrow Y_t \to +\infty$ if $Y_0 > 0$ AND $Y_t \to -\infty$ if $Y_0 < 0$

$\lambda < -1$ $\Rightarrow Y_t$ EXPLOSIVE OSCILLATIONS

2nd order DIFFERENCE EQUATION

$$Y_t = t_1 Y_{t-1} + t_2 Y_{t-2} + a + b X_t$$

(3) $$(1 - t_1 L - t_2 L^2) Y_t = a + b X_t$$

SOLUTION:

$$Y_t = \frac{a}{1 - t_1 - t_2} + \frac{b}{1 - t_1 L - t_2 L^2} X_t$$

(WE IGNORE FOR A WHILE THE TERM BEING SOLUTION OF $Y_t - t_1 Y_{t-1} - t_2 Y_{t-2} = 0$)

REWRITE THE POLYNOMIAL

$$1 - t_1 L - t_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L) = 1 - (\lambda_1 + \lambda_2) L + \lambda_1 \lambda_2 L^2$$

SO THAT

$$\boxed{\begin{aligned} \lambda_1 + \lambda_2 &= t_1 \\ -\lambda_1 \lambda_2 &= t_2 \end{aligned}}$$

COEFFICIENTS OF THE POLYNOMIAL WRITTEN IN TERMS OF λ.

WHAT are λ's?:

1° the equation: $1 - t_1 L - t_2 L^2$ has at most 2 roots: L_1 and L_2

2° $1 - t_1 L - t_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L) = \lambda_1 \lambda_2 \left(\frac{1}{\lambda_1} - L \right) \left(\frac{1}{\lambda_2} - L \right)$

SO THAT λ_1 and λ_2 ARE RECIPROCALS OF THE ROOTS OF THE AUTOREGRESSIVE POLYNOMIAL. $L_1 = \frac{1}{\lambda_1}$, $L_2 = \frac{1}{\lambda_2}$

THEY ARE THEMSELVES ROOTS OF THE FOLLOWING POLYNOMIAL:

$L^{-2} - t_1 L^{-1} - t_2 = 0 = z^2 - t_1 z - t_2 = 0$, where $z = \frac{1}{L}$

ALSO

$$(L_1, L_2) = \frac{t_1 \pm \sqrt{t_1^2 + 4 t_2}}{-2 t_2}$$

① WE FIRST FIND λ_1, λ_2:

ASSUME $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq 1$; REWRITE (3) AS:

$$(1 - \lambda_1 L)(1 - \lambda_2 L) Y_t = a + b X_t$$

$$Y_t = \frac{a}{(1-\lambda_1)(1-\lambda_2)} + \frac{b}{(1-\lambda_1 L)(1-\lambda_2 L)} X_t + c_1 \lambda_1^t + c_2 \lambda_2^t$$

c_1, c_2 ARE ARBITRARY CONSTANTS. AGAIN

$$(1 - \lambda_1 L)(1 - \lambda_2 L) c_1 \lambda_1^t = (1 - \lambda_1 L)(1 - \lambda_2 L) c_2 \lambda_2^t = 0$$

Use Y_0 to replace c_1 and c_2:

for $\lambda_1 \neq \lambda_2$:

$$\frac{1}{(1-\lambda_1 L)(1-\lambda_2 L)} = \frac{1}{\lambda_1 - \lambda_2}\left(\frac{\lambda_1}{1-\lambda_1 L} - \frac{\lambda_2}{1-\lambda_2 L}\right)$$

(3) CAN BE WRITTEN AS:

$$Y_t = \frac{a}{(1-\lambda_1)(1-\lambda_2)} + \frac{\lambda_1 b}{\lambda_1 - \lambda_2} \cdot \frac{1}{1-\lambda_1 L} X_t$$

$$- \frac{\lambda_2 b}{\lambda_1 - \lambda_2} \cdot \frac{1}{1-\lambda_2 L} X_t + c_1 \lambda_1^t + c_2 \lambda_2^t$$

$$Y_t = a \sum_{i=0}^{\infty} \lambda_1^i \sum_{j=0}^{\infty} \lambda_2^j + \frac{\lambda_1 b}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_1^i X_{t-i}$$

$$- \frac{\lambda_2 b}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_2^i X_{t-i} + c_1 \lambda_1^t + c_2 \lambda_2^t$$

THE FIRST TWO SUMS ARE FINITE $\Leftrightarrow$ $|\lambda_1| < 1$, $|\lambda_2| < 1$

SUPPOSE $a = 0$ FOR SIMPLICITY

$$Y_t = \frac{\lambda_1 b}{\lambda_1 - \lambda_2} \sum_{i=0}^{t-1} \lambda_1^i X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2} \sum_{i=0}^{t-1} \lambda_2^i X_{t-i}$$

$$+ \frac{\lambda_1 b}{\lambda_1 - \lambda_2} \sum_{i=t}^{\infty} \lambda_1^i X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2} \sum_{i=t}^{\infty} \lambda_2^i X_{t-i} + c_1 \lambda_1^t + c_2 \lambda_2^t$$

YIELDS

$$Y_t = \frac{\lambda_1 b}{\lambda_1 - \lambda_2} \sum_{i=0}^{t-1} \lambda_1^i X_{t-i} - \frac{\lambda_2 b}{\lambda_1 - \lambda_2} \lambda_2^i X_{t-i} + \lambda_1^t \theta_0 + \lambda_2^t \eta_0 , \quad t \geq 1$$

WHERE

$$\theta_0 = c_1 + \frac{b\lambda_1}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_1^i X_{0-i}$$

$$\eta_0 = c_2 - \frac{b\lambda_2}{\lambda_1 - \lambda_2} \sum_{i=0}^{\infty} \lambda_2^i X_{0-i}$$

WHEN $X_t = 0$

$$Y_t = \lambda_1^t \theta_0 + \lambda_2^t \eta_0 , \quad t \geq 1$$

- **IF $\theta_0 = \eta_0 = 0$**, $\Rightarrow Y_t = 0 \quad \forall t \geq 0$, HENCE $Y = 0$ IS STATIONARY PT, LONG RUN EQUILIBRIUM

- **λ_1, λ_2 REAL:**

$$\lim_{t \to \infty} Y_t = 0 \iff |\lambda_1| < 1 \text{ and } |\lambda_2| < 1$$

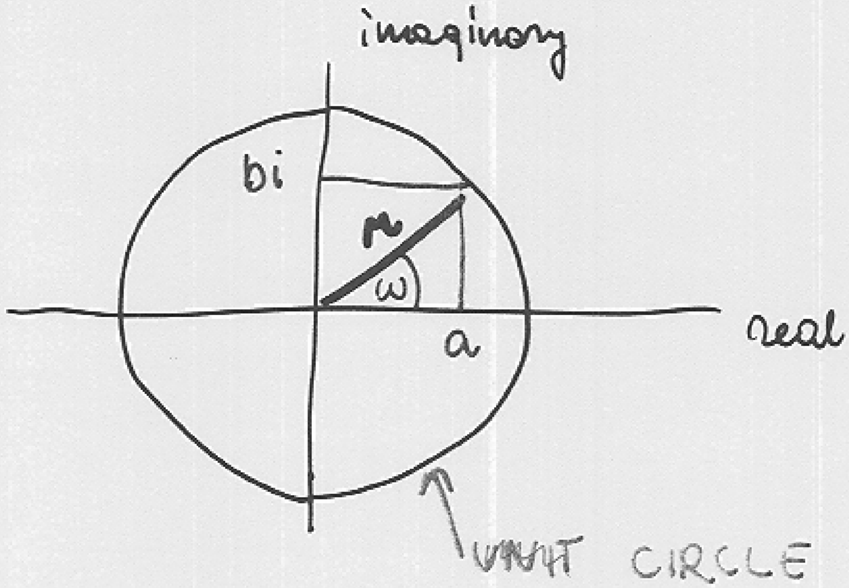
if: $\lambda_1 > 1$, $|\lambda_2| < |\lambda_1|$ AND $\theta_0 > 0 \Rightarrow \lim_{t \to \infty} Y_t = +\infty$

$\lambda_1 > 1$, $|\lambda_2| < |\lambda_1|$ AND $\theta_0 < 0 \Rightarrow \lim_{t \to \infty} Y_t = -\infty$

Y CONVERGES TOWARDS STATIONARY PT $\iff$

$$|\lambda_1| < 1 \quad \text{AND} \quad |\lambda_2| < 1$$

- λ_1, λ_2 COMPLEX



$a + bi = r\cos\omega + ri\sin\omega$

<u>Since</u>: $\cos\omega = \frac{a}{r}$

$\sin\omega = \frac{bi}{r}$

and

$= r(\cos\omega + i\sin\omega)$

$= r\left(\frac{a}{r} + i\ \frac{b}{r}\right) = a + bi$

$r = \sqrt{a^2 + b^2}$

$\Rightarrow$ OCCUR AS A COMPLEX CONJUGATE PAIR

$\lambda_1 = re^{i\omega} = r(\cos\omega + i\sin\omega)$

$\lambda_2 = re^{-i\omega} = r(\cos\omega - i\sin\omega)$

$r\cos\omega$ is the real part, $\pm r\sin\omega$ imaginary with $(i = \sqrt{-1})$

hence:

$$Y_t = \lambda_1^t \theta_0 + \lambda_2^t \eta_0$$

$$= (r e^{i\omega})^t \theta_0 + (r e^{-i\omega})^t \eta_0$$

$$= \theta_0 r^t [\cos \omega t + i \sin \omega t] + \eta_0 r^t [\cos \omega t - i \sin \omega t]$$

$$= (\theta_0 + \eta_0) r^t \cos \omega t \quad + i (\theta_0 - \eta_0) r^t \sin \omega t$$

$\Rightarrow$ **OSCILLATIONS!**

Since Y_t must be real, $(\theta_0 + \eta_0)$ must be real and $(\theta_0 - \eta_0)$ imaginary.

$\Rightarrow$ hence θ_0 and η_0 must be complex conjugates:

$$\theta_0 = p e^{i\theta}$$
$$\eta_0 = p e^{-i\theta}$$

$$Y_t = p e^{i\theta} r^t e^{i\omega t} + p e^{-i\theta} r^t e^{-i\omega t} =$$

$$p r^t [e^{i(\omega t + \theta)} + e^{-i(\omega t + \theta)}] = 2p r^t \cos(\omega t + \theta)$$

this is the solution of Y_t for $X_t = 0 \ \forall t$ and when roots are complex

Y_t OSCILATES WITH FREQUENCY DETERMINED BY ω

r^t IS THE DAMPING FACTOR, r IS AMPLITUDE OF COMP. ROOTS

$Y=0$ is STATIONARY PT APROACHED AS $t \to \infty \Leftrightarrow \underline{r<1}$!

- if $r>1$ $\quad Y_t$ EXPLOSIVE OSCILATIONS (unless $Y_0=0, Y_1=0 \ldots$)
- if $r<1$ $\quad$ DAMPED OSCILATION
- if $r=1$ $\quad$ PERIODIC OSCILLATIONS

what values of t_1, t_2 YIELD COMPLEX ROOTS?

$$\Delta < 0 \Leftrightarrow t_1^2 + 4t_2 < 0$$

$$\Leftrightarrow t_2 < 0$$

roots are reciprocals of:

$$\lambda_1 = \frac{t_1}{2} + \frac{i\sqrt{-(t_1^2+4t_2)}}{2} = a+bi = r\cos w + r\, i \sin w = $$
$$= re^{iw}$$

$$\lambda_2 = \frac{t_1}{2} - \frac{i\sqrt{-(t_1^2+4t_2)}}{2} = a-bi$$

Since $r = \sqrt{a^2+b^2}$:

$$r = \sqrt{\left(\frac{t_1}{2}\right)^2 - \frac{(t_1^2+4t_2)}{4}} = \sqrt{-t_2}$$

FOR OSCILATIONS TO BE DUMPED $r<1 \Rightarrow$

(1) $$\sqrt{-t_2} < 1 \Rightarrow t_2 > -1$$

note: $\cos w = \frac{a}{r} = \frac{t_1}{2\sqrt{-t_2}} \Rightarrow w = \cos^{-1}\left(\frac{t_1}{2\sqrt{-t_2}}\right)$

• RECALL FOR REAL λ_1, λ_2 WE REQUIRED $|\lambda_1|<1, |\lambda_2|<1$.

In terms of t_1, t_2 coefficients:

(1) $-1 < \frac{t_1 + \sqrt{t_1^2 + 4t_2}}{2} < 1$ ②

(2) $-1 < \frac{t_1 - \sqrt{t_1^2 + 4t_2}}{2} < 1$ ③

② $\Rightarrow$ $\frac{1}{2}\, t_1 + \sqrt{t_1^2 + 4t_2} < 1$

$t_1 + t_2 < 1$

③ $\Rightarrow$ $t_2 < 1 + t_1$

together:

1 $t_2 > -1$

2 $t_1 + t_2 < 1$

3 $t_2 - t_1 < 1$

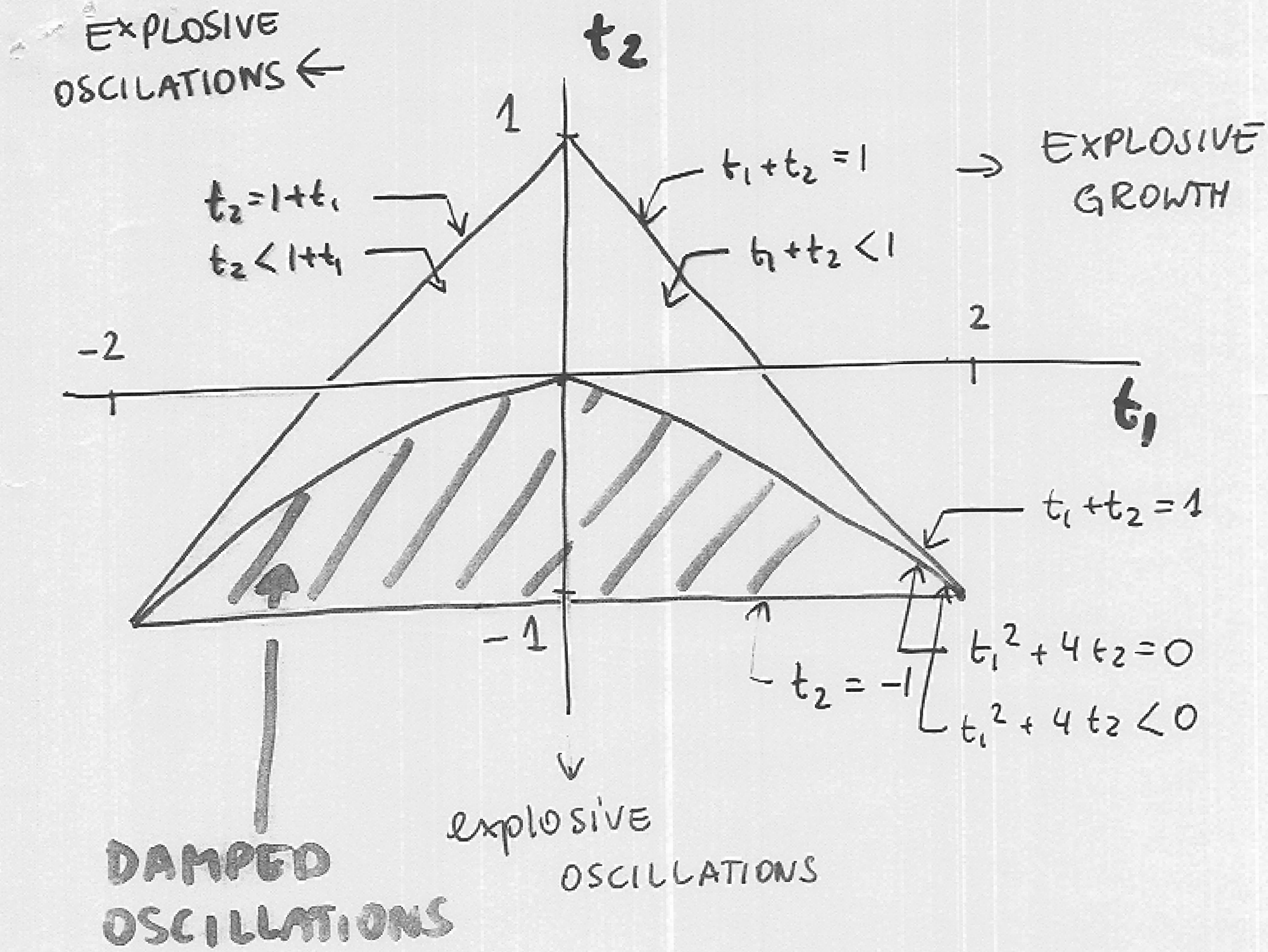
TO ACHIEVE STATIONARY BEHAVIOR $\Rightarrow$

CONVERGENCE

ROOTS OF THE AUTOREGRESSIVE POLYNOMIAL

MUST LIE OUTSIDE THE UNIT CIRCLE

$\Rightarrow$ $|\lambda_1|, |\lambda_2|$ their reciprocals are less than 1


EXPLOSIVE
OSCILLATIONS ←
t_2
1
$t_2 = 1 + t_1$
$t_2 < 1 + t_1$
$t_1 + t_2 = 1$
$t_1 + t_2 < 1$
→ EXPLOSIVE
GROWTH
−2
2
t_1
$t_1 + t_2 = 1$
−1
$t_1^2 + 4t_2 = 0$
$t_2 = -1$
$t_1^2 + 4t_2 < 0$
explosive
OSCILLATIONS
DAMPED
OSCILLATIONS