MULTIVARIATE PROCESSES

WOLD THEOREM:

ANY 2nd order stationary Y_t such that $LE(Y_t \mid Y_{t-h})$ tends to the marginal mean when $h \to \infty$ has an ∞ VMA representation:

$$Y_t = \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \Theta_2 \varepsilon_{t-2} + \dots$$

where Θ are $n \times n$, ε_t is an n-dim. white noise $\equiv$ linear innovation of Y_t : $\varepsilon_t = Y_t - LE(Y_t \mid Y_{t-h})$

- VAR(p)

$$Y_t = \Phi_1 Y_{t-1} + \dots + \Phi_p Y_{t-p} + \varepsilon_t$$

- VMA(q)

$$Y_t = \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \dots + \Theta_q \varepsilon_{t-q}$$

- VARMA(p,q)

$$Y_t = \Phi_1 Y_{t-1} + \dots + \Phi_p Y_{t-p} + \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \dots + \Theta_q \varepsilon_{t-q}$$

$$\Phi(L)\, Y_t = \Theta(L)\, \varepsilon_t$$

$$\Phi(L) = Id - \Phi_1 L - \dots - \Phi_p L^p$$

$$\Theta(L) = Id - \Theta_1 L - \dots - \Theta_q L^q$$

- Stationarity / invertibility:

  - FOR $\Phi(L)$: <u>stationary</u> if roots of characteristic equation

$$\det \Phi(x) = 0$$

strictly larger than 1 in abs. value.
(eigenvalues of modulus <1)

  - FOR $\Theta(L)$: invertibility: roots of $\det \Theta(x) = 0$ greater than 1 in abs value.

VAR:

STRUCTURAL:

$$\begin{cases} y_t = b_{10} - b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{y_t} \\ z_t = b_{20} - b_{21} y_t + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{z_t} \end{cases}$$

y_t, z_t stationary. $[\varepsilon_{y_t}, \varepsilon_{z_t}]'$ W.N with $\Omega = \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_z^2 \end{bmatrix}$.
i.e. are uncorrelated.

$$\begin{bmatrix} 1 & b_{12} \\ b_{21} & 1 \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{y_t} \\ \varepsilon_{z_t} \end{bmatrix}$$

$$B x_t = \Gamma_0 + \Gamma_1 x_{t-1} + \varepsilon_t$$

STANDARD:

$$\underline{x_t = A_0 + A_1 x_{t-1} + e_t}$$

$A_0 = B^{-1}\Gamma_0$, $A_1 = B^{-1}\Gamma_1$, $e_t = B^{-1}\varepsilon$

STANDARD

$$y_t = a_{10} + a_{11}\, y_{t-1} + a_{12}\, z_{t-1} + e_{1t}$$

$$z_t = a_{20} + a_{21}\, y_{t-1} + a_{22}\, z_{t-1} + e_{2t}$$

new shocks:

$$e_{1t} = (\varepsilon_{yt} - b_{12}\,\varepsilon_{zt}) / (1 - b_{12} b_{21})$$

$$e_{2t} = (\varepsilon_{zt} - b_{21}\,\varepsilon_{yt}) / (1 - b_{12} b_{21})$$

serially uncorrelated

contempran. correlation:

$$E\left[e_{1t}\, e_{2t}\right] = - b_{21}(\sigma_y^2 - b_{12}\,\sigma_z^2) / (1 - b_{12} b_{21})$$

Hence:

$$\Sigma = \begin{bmatrix} var(e_{1t}) & cov(e_{1t}\, e_{2t}) \\ cov(e_{12}\, e_{2t}) & var(e_{2t}) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

Note: one can orthogonalize the shocks using the Cholesky decomposition:

$$\Sigma = RR'$$

The new orthogonalized errors $u_t = R^{-1} e_t$ satisfy:

$$RR' = \Sigma \Rightarrow R^{-1} \Sigma R^{-1'} \Rightarrow R^{-1}\left[E\, e_t e_t'\right] R^{-1'} = Id \Rightarrow$$

$$\Rightarrow E\left[u_t\, u_t'\right] = Id$$

stationarity:

$$y_t = \left[a_{10}(1-a_{22}) + a_{12}a_{20} + (1-a_{22}L)e_{1t} + a_{12}e_{2t-1}\right]/\Phi$$

$$z_t = \left[a_{20}(1-a_{11}) + a_{21}a_{10} + (1-a_{11})L)e_{2t} + a_{21}e_{1t-1}\right]/\Phi$$

$$\Phi = (1-a_{11}L)(1-a_{22}L) - a_{12}a_{21}L^2$$ $\Rightarrow$ roots of this polynomial outside the unit circle.

ESTIMATION / IDENTIFICATION.

• STANDARD VAR:

CAN BE ESTIMATED BY OLS EQUATION BY EQUATION,

$\Rightarrow$ CONSISTENT AND EFFICIENT (See SUR)

when all regressors are the same in both equations.

BUT YOU WILL NOT BE ABLE TO RECOVER THE PARAMETERS OF THE STRUCTURAL VAR.

you get: $\hat{a}_{10}, \hat{a}_{20}, \hat{a}_{12}, \hat{a}_{21}, \hat{a}_{22}, \hat{a}_{4}, \hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\sigma}_{12}$ 9 coeff

structural has 10 coefficients!

$\Rightarrow$ underidentified.

have to restrict 1 parameter to get an exactly identified system

• RECURSIVE STRUCTURE IS ASSUMED:

$b_{21} = 0$

$$
\begin{aligned}
y_t &= b_{10} - b_{12} z_t + \gamma_{11} y_{t-1} + \gamma_{12} z_{t-1} + \varepsilon_{yt} \\
z_t &= b_{20} \qquad\qquad + \gamma_{21} y_{t-1} + \gamma_{22} z_{t-1} + \varepsilon_{zt}
\end{aligned}
$$

$$
B^{-1} = \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \quad \text{SINCE} \quad B = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}
$$

premultiply by B^{-1}

$$
\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_{10} \\ b_{20} \end{bmatrix} + \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & -b_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}
$$

$$
\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} b_{10} - b_{12} b_{20} \\ b_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} - b_{12}\gamma_{21} & \gamma_{21} - b_{12}\gamma_{22} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} - b_{12}\varepsilon \\ \varepsilon_{zt} \end{bmatrix}
$$

recursive.

$$
\begin{aligned}
y_t &= a_{10} + a_{11} y_{t-1} + a_{12} z_{t-1} + e_{1t} \\
y_t &= a_{20} + a_{21} y_{t-1} + a_{22} z_{t-1} + e_{2t}
\end{aligned}
$$

$$
\left.
\begin{array}{l}
\bullet\ e_{1t} = \varepsilon_{yt} - b_{12} \varepsilon_{zt} \\
\bullet\ e_{2t} = \varepsilon_{zt} \\
\quad \mathrm{var}(e_1 t) = \sigma_y^2 + b_{12}^2 \sigma_z^2 \\
\quad \mathrm{var}(e_{2t}) = \sigma_z^2, \quad \mathrm{cov}(e_1 e_2) = -b_{12}\sigma_z^2
\end{array}
\right\} \text{Choleski decomposition}
$$

IMPULSE RESPONSE

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{10} \\ a_{20} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

VMA(∞):

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \bar{y} \\ \bar{z} \end{bmatrix} + \sum_{i=0}^{\infty} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^i \begin{bmatrix} e_{1,t-i} \\ e_{2,t-i} \end{bmatrix}$$

recall

$$\begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} = \frac{1}{1-b_{12}b_{21}} \begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

substitute to get IM. RES in terms of structural shocks:

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \bar{y}_t \\ \bar{z}_t \end{bmatrix} + \frac{1}{1-b_{12}b_{21}} \sum_{i=0}^{\infty} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^i \begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

define ϕ_i a 2×2 with $\phi_{jk}(i)$:

$$\phi_i = A_1^i \frac{1}{1-b_{12}b_{21}} \begin{bmatrix} 1 & -b_{12} \\ -b_{21} & 1 \end{bmatrix}$$

$$\begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \bar{y}_t \\ \bar{z} \end{bmatrix} + \sum_{i=0}^{\infty} \begin{bmatrix} \phi_{11}(i) & \phi_{12}(i) \\ \phi_{21}(i) & \phi_{22}(i) \end{bmatrix} \begin{bmatrix} \varepsilon_{y\,t-i} \\ \varepsilon_{z\,t-i} \end{bmatrix}$$

$$X_t = \mu + \sum_{i=0}^{\infty} \phi_i \, \varepsilon_{t-i}$$

ϕ_{jk} : impact multipliers:

$\phi_{12}(0)$: INSTANTANEOUS IMPACT OF A ONE UNIT CHANGE in ε_{zt} on y

$\phi_{11}(i)$, $\phi_{12}(i)$ ... <u>impulse response functions.</u>

Long run multiplier : cumulated sum of the effects of ε_{zt} on $\{y_t$

$$= \sum_{i=0}^{n} \phi_{12}(i)$$

by stationarity $\sum_{i=0}^{\infty} \phi_{jk}^2 < \infty$.

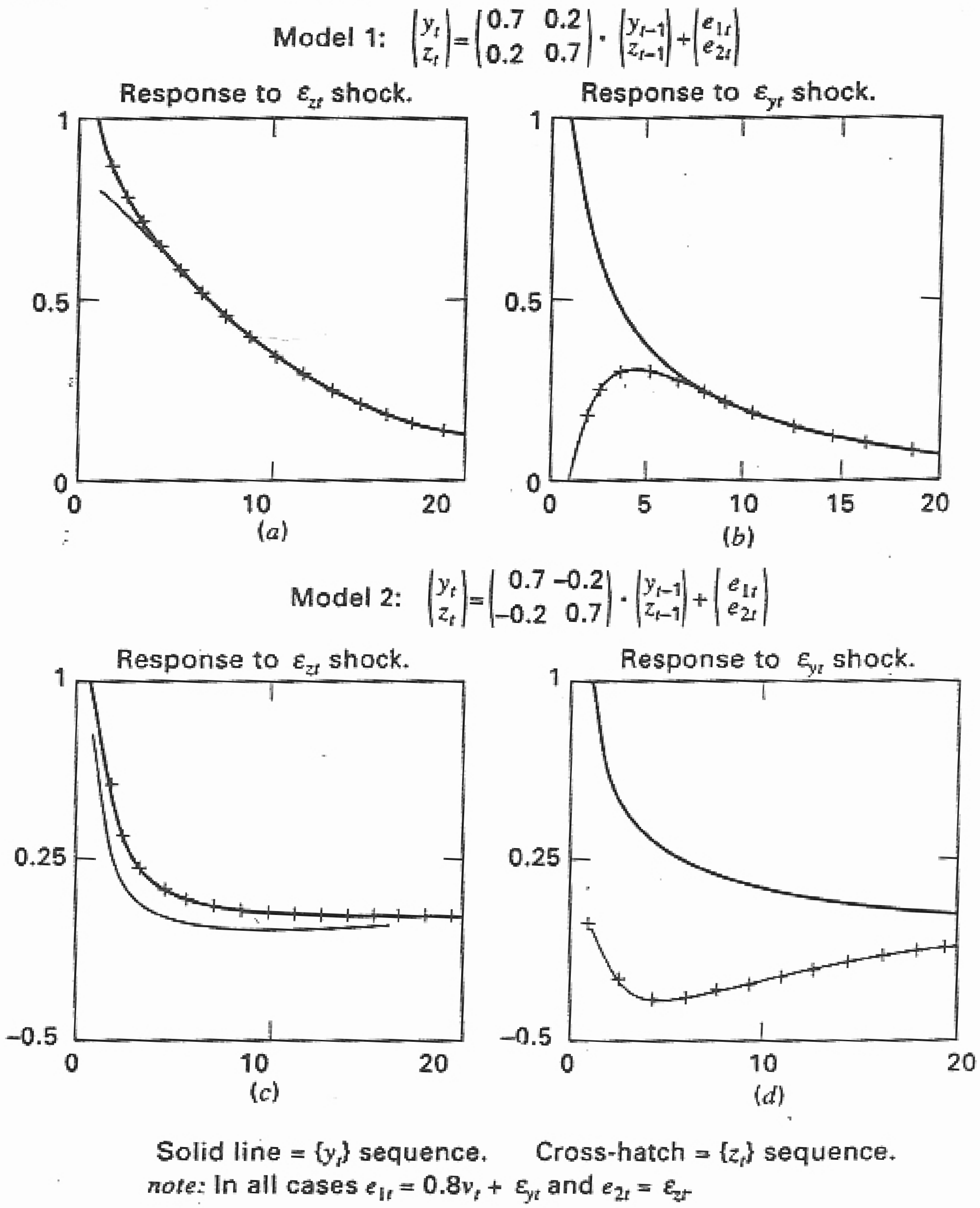
IN PRACTICE : RESTRICT $b_{21} = 0$

$e_{1t} = \varepsilon_{yt} - b_{12}\, \varepsilon_{zt}$

$e_{2t} = \varepsilon_{zt}$

ε_{zt} affects directly e_{1t}, e_{2t}, but ε_{yt} only e_{1t} : z_t "leads" the system.

Figure 5.6 Two impulse reponse functions.



the subsequent values of the $\{y_t\}$ and $\{z_t\}$ sequences converge to their long-run levels. This convergence is assured by the stability of the system; as found earlier, the two characteristic roots are 0.5 and 0.9.

The effects of a one-unit shock in ϵ_{yt} are shown in the upper-right-hand graph (b) of the figure. The asymmetry of the decomposition is immediately seen by comparing the two upper graphs. A one-unit shock in ϵ_{yt} causes the value of y_t to increase by one unit; however, *there is no contemporaneous effect on the value of* z_t, so that $y_t = 1$ and $z_t = 0$. In the subsequent period, ϵ_{yt+1} returns to zero. The autoregressive nature of the system is such that $y_{t+1} = 0.7y_t + 0.2z_t = 0.7$ and $z_{t+1} = 0.2y_t + 0.7z_t = 0.2$. The remaining points in the figure are the impulse responses for periods y_{t+2}

VARIANCE DECOMPOSITION

use the VMA(∞)

$$x_{t+n} = \mu + \sum_{i=0}^{\infty} \phi_i \, \varepsilon_{t+n-i}$$

n-period forecast error:

$$x_{t+n} - E_t \, x_{t+n} = \sum_{i=0}^{n-1} \phi_i \, \varepsilon_{t+n-i}$$

- Select $\{y_t\}$:

$$y_{t+n} - E_t \, y_{t+n} = \phi_{11}(0)\, \varepsilon_{y\,t+n} + \phi_{11}(1)\, \varepsilon_{y,\,t+n-1} + \dots + \phi_{11}(n-1)\, \varepsilon_{y\,t+1}$$

$$+ \phi_{12}(0)\, \varepsilon_{z\,t+n} + \phi_{12}\, \varepsilon_{z\,t+n-1} + \dots + \phi_{12}(n-1)\, \varepsilon_{z\,t+1}$$

denote the variance of n-step ahead forecast error ~~variance~~ of y_{t+n} as $\sigma_y(n)^2$:

$$\sigma_y(n)^2 = \sigma_y^2 \left[\phi_{11}(0)^2 + \phi_{11}(1)^2 + \dots + \phi_{11}(n-1)^2 \right]$$

$$+ \sigma_z^2 \left[\phi_{12}(0)^2 + \phi_{12}(1)^2 + \dots + \phi_{12}(n-1)^2 \right]$$

variance of the forecast error ↑ when n ↑

decompose the n-step error variance due to each shock

the proportion of $\sigma_y(n)^2$ due to:

1) ε_{yt} $\Rightarrow$ $\sigma_y^2 [\phi_{11}(0)^2 + \phi_{11}(1)^2 + \dots + \phi_{11}(n-1)^2] / \sigma_y(n)^2$

2) ε_{zt} $\Rightarrow$ $\sigma_{zz}^2 [\phi_{12}(0)^2 + \phi_{12}(1)^2 + \dots + \phi_{12}(n-1)^2] / \sigma_y(n)^2$

$\{y_t\}$ is exogenous if ε_{zt} explain none of the forecast error variance of $\{y_t\}$ at all forecast horizons.

HYPOTHESIS TESTING.

$$\begin{bmatrix} x_{1t} \\ \vdots \\ x_{nt} \end{bmatrix} = \begin{bmatrix} A_{10} \\ \vdots \\ A_{n0} \end{bmatrix} + \begin{bmatrix} A_{11}(L) & \cdots & A_{1n}(L) \\ \vdots & & \\ A_{n1}(L) & & A_{nn}(L) \end{bmatrix} \begin{bmatrix} x_{1t-1} \\ \vdots \\ x_{nt-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ \vdots \\ e_{nt} \end{bmatrix}$$

if lag length differs across equations: near VAR: use full SUR approach.

if lag length $= p$, each of n equations has $np+1$ parameters to chose the correct p:

1. estimate the longest VAR, compute Σ, say $p=12 \Rightarrow \Sigma_{12}$
2. to check if $p=8$ is correct:

reestimate with $p=8 \Rightarrow \Sigma_8$

LR: $$T(\log|\Sigma_8| - \log|\Sigma_{12}|) \sim \chi^2(2)$$

or $(T-c)$

H_0: lag length $p=8$

if LR stat is large reject $H_0: p=8$.

in general: $$(T-c)(\log|\Sigma_2| - \log|\Sigma_u|)$$

use AIC $= T\log|\Sigma| + 2N$

SBS $= T\log|\Sigma| + N\log(T)$

Structural VAR's.

identification: Σ has $(n^2+n)/2$ elements.

B has ones on the main diag: n^2-n elements + n unknown

$var(\varepsilon_{it}) \Rightarrow n^2$ elements

to identify n^2 unknowns from $\frac{n^2+n}{2}$ elements of Σ needs

$$n^2 - [(n^2+n)/2] = (n^2-n)/2 \text{ restrictions.}$$