COINTEGRATION

A Linear combination of integrated variables is stationary $\Rightarrow$ COINTEGRATED VARIABLES

$\Rightarrow$ an equilibrium condition for nonstationary variables

$\Rightarrow$ their stochastic trends must be linked

can't move independently.

ex. of a long-run equilibrium:

$$\beta_1 x_{1t} + \beta_2 x_{2t} + \dots + \beta_n x_{nt} = 0$$

$\beta = [\beta_1 \ \beta_2 \ \dots \ \beta_n]$, $x_t = [x_{1t}, x_{2t} \dots x_{nt}]$

the system is in equilibrium when

$$\beta x_t = 0$$

- the deviation from long-run equilibrium = equilibrium error

$$e_t = \beta x_t$$

has to be stationary (reverting to zero) [no permanent deviati from equilibrium]

DEFINITION (Engle, Granger)

the components of the vector $x_t = [x_{1t}, x_{2t} \dots x_{nt}]$ are cointeg of order d, b, $\underline{x_t \sim CI(d, b)}$ if

- all components of x_t are integrated of order d
- there exists $\beta = [\beta_1, \beta_2 \dots \beta_n]$ such that βx_t is integrated of order $(d-b)$, where $b > 0$. β is the COINTEGRATING VECTOR

Remarks

1. A linear combination

β is not unique : if $(\beta_1, \beta_2 \dots \beta_n)$ is a cointegrating vector, then for any $\lambda \neq 0$ $[\lambda\beta_1, \lambda\beta_2, \dots \lambda\beta_n]$ is also a cointeg. vector.

We **normalize** the cointegrating vector by fixing $\beta_1 = 1$, i.e. select $\lambda = \frac{1}{\beta_1}$ yielding $\left[1, \frac{\beta_2}{\beta_1}, \dots \frac{\beta_n}{\beta_1}\right]$

2. All variables must be integrated of the same order, $I(1)$ for ex. [shocks dissipate at the same rate]

3. if x_t has n components, there may be **n-1** linearly independent cointegrating vectors.

   if x_t has 2 components : at most one cointeg. vector.

   the # of cointegrating vectors = cointegrating rank of x_t

4. Most often $CI(1,1)$

   macro data or finan. data have 1 unit root.

Practical (2 variables)

Scatterplot : plot y_t against z_t. If there is a straight line that fits to the cluster, the regression coefficients form the cointegrating vector.

- the residuals from this regression have to be stationary.

"spurious regressions revisited"

Idea: trend in one variable is a linear combination of the trends in other variables.

ex:
$$y_t = \mu_{yt} + \varepsilon_{yt}$$
$$z_t = \mu_{zt} + \varepsilon_{zt}$$

where
μ_{it} : is a random walk representing the trend
ε_{it} : stationary (irregular) component.

$$\underbrace{\beta_1 y_t + \beta_2 z_t}_{\text{stationary}} = \beta_1(\mu_{yt} + \varepsilon_{yt}) + \beta_2(\mu_{zt} + \varepsilon_{zt})$$
$$= \underbrace{(\beta_1 \mu_{yt} + \beta_2 \mu_{zt})}_{\downarrow\ \text{MUST VANISH}} + \underbrace{(\beta_1 \varepsilon_{yt} + \beta_2 \varepsilon_{zt})}_{\text{stationary}}$$

$\Rightarrow$ necessary and sufficient condition

$$\beta_1 \mu_{yt} + \beta_2 \mu_{zt} = 0$$

i.e. $\mu_{yt} = -\dfrac{\beta_2}{\beta_1}\mu_{zt}$

up to the scalar $-\dfrac{\beta_2}{\beta_1}$ the two $I(1)$ processes must have the same stochastic trend to be $CI(1,1)$

$\Rightarrow \quad \mu_{yt} = \mu_{zt} = \mu_t \quad \Rightarrow \beta = [1, -1]$

will purge the trend from linear combination.

In a multivariate case of n variables $x_t = [x_{1t} \dots x_{nt}]$

$$x_t = \mu_t + \varepsilon_t$$

$$\beta \mu_t = 0$$

and $\beta x_t = \beta \varepsilon_t$, is stationary.

if the cointegrating rank is r, there is $r<n$ linear combinations btw the trends. : $\beta = (r \times n)$

ERROR CORRECTION

the paths of variables are influenced by the extent of any deviation from long-run equilibrium.

if the system is to return to equilibrium, at least one variable must respond to the magnitude of the deviation.

ex: r_{Lt} and r_{St}: if the gap btw them is "large", relative to the long-run equilibrium, the short rate must rise to the long-term rate.

The gap has to be closed $\Rightarrow$ there is a short run dynamics, influenced by the deviation from the long run equilibrium.

Error correction model:

r_{Lt}, r_{St} long and short term interest rates, each $I(1)$

$$\Delta r_{St} = \alpha_S (r_{L,t-1} - \beta r_{S,t-1}) + \varepsilon_{St} \quad , \alpha_S > 0$$

$$\Delta r_{Lt} = -\alpha_L (r_{L,t-1} - \beta r_{S,t-1}) + \varepsilon_{Lt} \quad , \alpha_L > 0$$

ε_{St} and ε_{Lt} are noises, can be correlated.

α_S, α_L, β : positive parameters.

$$\boxed{\begin{matrix}\text{Stat}\\ I(0)\end{matrix}} = \underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{?}$$

$\left.\begin{matrix}\Delta r_{St}\\ \Delta r_{Lt}\end{matrix}\right\}$ respond to 1) ε's and 2) $(r_{Lt-1} - \beta r_{S,t-1})$

previous period's deviation from long run equilibrium.

the r.h.s is $I(0)$ if $(r_{Lt-1} - \beta r_{S,t-1})$ is $I(0)$, i.e. $r_{L,t}$ and $r_{S,t}$ are cointegrated with the vector

$$[1, -\beta]$$

$\Rightarrow$ r_{Lt} and r_{St} are $C(1,1)$

A more general model with more lags:

$$\Delta r_{S,t} = a_{10} + \alpha_S (r_{L,t-1} - \beta r_{S,t-1}) + \sum a_{11}(i) \Delta r_{S,t-i} + \sum a_{12}(i) \Delta r_{L,t-i} + \varepsilon_{S,t}$$

$$\underbrace{\Delta r_{L,t}}_{\text{stationary}} = a_{20} - \alpha_L (r_{L,t-1} - \beta r_{S,t-1}) + \sum a_{21}(i) \Delta r_{S,t-i} + \sum a_{22}(i) \Delta r_{L,t-i} + \varepsilon_{L,t}$$

$\Rightarrow$ VAR in first differences augmented by the error correction

terms : $\alpha_S (r_{L,t-1} - \beta r_{S,t-1})$

$-\alpha_L (r_{L,t-1} - \beta r_{S,t-1})$

α_S, α_L: SPEED OF ADJUSTMENT PARAMETERS.

- the larger α_S $\Rightarrow$ the greater the response of r_S to deviation

IF $\alpha_S = 0$ and $a_{12}(i) = 0$ $\Rightarrow$ $r_{S,t}$ is unresponsive to the last period deviation.

ABSENCE OF GRANGER CAUSALITY:

SAME TEST OF $a_{12}(i) = 0$ AND ADDITIONALLY THAT THE SPEED OF ADJUSTMENT COEFFICIENT $\alpha_S = 0$.

IN ECM.

Consider n variables model

$x_t = [x_{1t}, x_{2t} \dots x_{n,t}]$ has an ECM:

$$\Delta x_t = \pi_0 + \underbrace{\pi x_{t-1}} + \pi_1 \Delta x_{t-1} + \pi_2 \Delta x_{t-2} + \dots$$

$$+ \dots \pi_p \Delta x_{t-p} + \varepsilon_t$$

π_0: $(n \times 1)$ vector of intercepts π_{i0}

π_i: $(n \times n)$ coeff matrix with $\pi_{jk}(i)$

π : matrix with elements π_{jk} such that one or more

$\pi_{jk} \neq 0$

ε_t $(n \times 1)$ vector of ε_{it}

ε_{it} may be correlated with ε_{jt}

- $$\underbrace{\pi x_{t-1}}_{\text{has to be stationary}} = \underbrace{\Delta x_t - \pi_0 - \sum \pi_i \Delta x_{t-i} - \varepsilon_t}_{\text{Stationary}}$$

$\Rightarrow$

π contains in each row a cointegrating vector of x_t

ex: $\pi_{11} x_{1t-1} + \pi_{12} x_{2,t-1} + \dots + \pi_{1n} x_{n,t-1}$ (1st row)

$[\pi_{11}, \pi_{12} \dots, \pi_{1n}]$ = COINTEGRATING VECTOR OF X

- IF Π = matrix of zeros $\Rightarrow$ ECM = VAR in 1st differences (no error correction) Δx_t does not respond to previous deviation
- if one or more $\pi_{jk} \neq 0$: Δx_t responds to the previous period deviation.

$\rightarrow$ estimation as a VAR in 1st differences is wrong:
$\rightarrow$ misspecification.

ex:
$$y_t = a_{11} y_{t-1} + a_{12} z_{t-1} + \varepsilon_{yt}$$
$$z_t = a_{21} y_{t-1} + a_{22} z_{t-1} + \varepsilon_{zt}$$

$$\begin{bmatrix} (1-a_{11}L) & -a_{12}L \\ -a_{21}L & (1-a_{22}L) \end{bmatrix} \begin{bmatrix} y_t \\ z_t \end{bmatrix} = \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

Solve the characteristic equation
$$(1-a_{11}L)(1-a_{22}L) - a_{12}a_{21}L^2 = 0$$

and set $\lambda_i = \frac{1}{L_i}$ $\quad i = 1, 2.$

1) if L_1, L_2 outside the unit circle $\Rightarrow \lambda_1, \lambda_2$ inside unit circle. variables are stationary, hence no cointegration.

2. if either λ_i outside unit circle $\Rightarrow$ explosive solution of one variable $\Rightarrow$ no coint. Also both $\lambda_i = 1$ excluded because they evolve without L.R. equilibrium

3. y_t and z_t are CI (1,1) when:

   one $\lambda_i = 1$

   other $|\lambda_i| < 1$.

example $\lambda_1 = 1$, then a univariate representation of y_t:

$$y_t = \frac{(1 - a_{22} L)\, \varepsilon_{yt} + a_{12} L\, \varepsilon_{zt}}{\phi}$$

where $\phi = (1 - a_{11} L)(1 - a_{22} L) - a_{12} a_{21} L^2$

becomes

$$y_t = \left[(1 - a_{22} L)\, \varepsilon_{yt} + a_{12} L\, \varepsilon_{zt} \right] / \left[(1 - L)(1 - \lambda_2 L) \right]$$

$$(1 - L)\, y_t = \Delta y_t = \left[(1 - a_{22} L)\, \varepsilon_{yt} + a_{12} L\, \varepsilon_{zt} \right] / (1 - \lambda_2 L)$$

is stationary when $|\lambda_2| < 1$.

The larger of the two λ_i's is unity when:

$$0.5 (a_{11} + a_{22}) + 0.5 \left[(a_{11}^2 + a_{22}^2) - 2 a_{11} a_{22} + 4 a_{12} a_{21} \right]^{\frac{1}{2}} = 1$$

$\Rightarrow$ • $a_{11} = [(1 - a_{22}) - a_{12} a_{21}] / (1 - a_{22})$

∴ since a_{12} and or a_{21} must differ from zero ⇒

- $|\lambda_2| < 1$ when $a_{22} > -1$
  - and $a_{12} a_{21} + (a_{22})^2 < 1$

3 restrictions on coeff to ensure $C(1,1)$

to see the consequences of these 3 restrictions:

$$\begin{bmatrix} \Delta y_t \\ \Delta z_t \end{bmatrix} = \begin{bmatrix} (a_{11}-1) & a_{12} \\ a_{21} & (a_{22}-1) \end{bmatrix} \begin{bmatrix} y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{yt} \\ \varepsilon_{zt} \end{bmatrix}$$

the first restriction on a_{11} ⇒ $(a_{11}-1) = -a_{12} a_{21} / (1-a_{22})$

$$\Delta y_t = -\left[a_{12} a_{21} / (1-a_{22})\right] y_{t-1} + a_{12} z_{t-1} + \varepsilon_{yt}$$

$$\Delta z_t = a_{21} y_{t-1} - (1-a_{22}) z_{t-1} + \varepsilon_{zt}$$

set $\alpha_y = -a_{12} a_{21} / (1-a_{22})$

$\beta = (1-a_{22})/a_{21}$

$\alpha_z = a_{21}$

$$\Delta y_t = \alpha_y (y_{t-1} - \beta z_{t-1}) + \varepsilon_{yt}$$

$$\Delta z_t = \alpha_z (y_{t-1} - \beta z_{t-1}) + \varepsilon_{zt}$$

The conditions:

$$\begin{cases} a_{11} = \left[(1-a_{22}) - a_{12}a_{21}\right] / (1-a_{22}) \\ a_{22} > -1 \\ a_{12}a_{21} + (a_{22})^2 < 1 \end{cases}$$

ensure that y_t and z_t are $C(1,1)$

- they guarantee that an ECM exists

Granger representation theorem:

For any set of $I(1)$ variables, error correction and cointegration are equivalent representations.

- Cointegration requires restrictions on coeff in a VAR

$$\Delta X_t = \Pi X_{t-1} + \varepsilon_t$$

it is wrong to estimate a VAR of cointegrated variables using first differences ⇒ this eliminates ΠX_{t-1}: the ~~error~~ error correction

- rows of Π are NOT LINEARLY INDEPENDENT IF VARIABLES ARE COINTEGRATED

1) determinant of $\Pi = 0$ when y_t, z_t cointegrated.

$\Rightarrow$ use the rank of Π to determine whether y_t, z_t cointegrated.

- in our system if $\lambda_1 = 1 \Rightarrow \det(\Pi) = 0$, <u>$\Pi$ has rank $= 1$</u>

- if <u>Π has rank 0</u> $\Rightarrow a_{11} = 1 = a_{22}$, $a_{12} = a_{21} = 0$

  y_t and z_t satisfy a stationary VAR in 1st differences.

- if <u>Π has a full rank</u> neither λ_i can be unity, y_t, z_t are jointly stationary.

z_t does not Granger cause y_t [illegible] and Δz_{t-i} do not enter the Δy_t equation [illegible] y_t does not respond to the deviation from long-run equilibrium.

JOHANSEN TEST OF COINTEGRATION

$$x_t = A_1 x_{t-1} + A_2 x_{t-2} + \dots + A_p x_{t-p} + \varepsilon_t$$

$$\Delta x_t = (A_1 - 1) x_{t-1} + A_2 x_{t-2} + \dots A_p x_{t-p} + \varepsilon_t$$

add/subtract $(A_1 - 1) x_{t-2}$:

$$\Delta x_t = (A_1 - 1)\Delta x_{t-1} + (A_2 + A_1 - 1) x_{t-2} + A_3 x_{t-3} + \dots A_p x_{t-p} + \varepsilon_t$$

add/subtract $(A_2 + A_1 - 1) x_{t-3}$:

$$\Delta x_t = (A_1 - 1)\Delta x_{t-1} + (A_2 + A_1 - 1)\Delta x_{t-2} + (A_3 + A_2 + A_1 - 1) x_{t-3} + \dots + A_p x_{t-p} + \varepsilon_t$$

$$\Delta x_t = \sum_{i=1}^{p-1} \Pi_i \, \Delta x_{t-i} + \Pi x_{t-1} + \varepsilon_t$$

rank of Π = # of eigenvalues $\neq 0$

rank them: $\lambda_1 > \lambda_2 \dots > \lambda_n$

- if variables in x_t NOT COINTEGRATED, $\text{rank}(\Pi) = 0$, all eigenvalues $= 0$

since $\ln(1) = 0$, we have $\ln(1 - \lambda_i) = 0$

- if $\text{rank}(\Pi) = 1 \Rightarrow 0 < \lambda_1 < 1$, so that $\ln(1 - \lambda_1) < 0$,

  all other $\lambda_i = 0 \Rightarrow \ln(1 - \lambda_2) = \ln(1 - \lambda_3) = \dots = 0$

compute

(1) $$\lambda_{trace} = -T \sum_{i=r+1}^{n} \ln(1 - \hat{\lambda}_i)$$

(2) $$\lambda_{max}(r, r+1) = -T \ln(1 - \hat{\lambda}_{r+1})$$

(1) H_0: # of distinct cointegrating vectors is less or equal r.

$\lambda_{trace} = 0$ when all $\lambda_i = 0$. [H_A general]

(2) H_0 : # of cointegrating vectors is r

H_A : # of cointegrating vectors is $(r+1)$ Johansen, Juselius (1990)