SELECTED PROPERTIES OF MOMENTS OF R.V.

X, Y random variables in $\mathbb{R}$ and r, s positive real numbers.

1. $E(|X|)$ is a real non-negative number or $E(|X|) = \infty$

2. $E(X)$ Exists & is finite $\Leftrightarrow E(|X|) < \infty$

3. $E(|X|) < \infty \Rightarrow |E(X)| \leq E(|X|) < \infty$

4. if $0 < r \leq s$ then $E(|X|^s) < \infty \Rightarrow E(|X|^r) < \infty$

5. $L_s \supseteq L_r$ for $0 < r \leq s$

6. $E(|X|^r) < \infty \Rightarrow E(X^K)$ exists & is finite for any integer $K : 0 < K \leq r$

7. a, b real numbers:

   $X \in L_r$ and $Y \in L_r \Rightarrow aX + bY \in L_r$

8. Hölder inequality : $r > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$

$$E(|XY|) \leq \left[E(|X|^r)\right]^{\frac{1}{r}} \left[E(|Y|^s)\right]^{\frac{1}{s}}$$

9. Cauchy-Schwarz inequality

$$E(|XY|) \leq \left[E(X^2)\right]^{\frac{1}{2}} \left[E(Y^2)\right]^{\frac{1}{2}}$$

10. JENSEN inequality

$g(x)$ a convex function on $\mathbb{R}$ and $E(|X|) < \infty$

then $$g(EX) \leq E\left[g(X)\right]$$

CONVERGENCE

$X_n : n = 1, 2 \ldots$ a sequence of real valued random variables on $(\Omega, \mathcal{A}, P)$ and X another real random variable on the same space $(\Omega, \mathcal{A}, P)$

1. X_n converges almost surely (a.s.) to X: $X_n \xrightarrow{a.s.} X$

iff $$P\left[\lim_{n \to \infty} X_n = X\right] = 1$$

2. X_n converges in probability to X: $X_n \xrightarrow{P} X$

iff $$\lim_{n \to \infty} P\left[|X_n - X| > \varepsilon\right] = 0, \quad \forall \varepsilon > 0$$

3. Suppose $E|X_n|^r < \infty$, $\forall n$ where $r > 0$

X_n converges in mean of order r to X: $X_n \xrightarrow{r} X$

iff $\lim\limits_{n\to\infty} E\left(|X_n - X|^r\right) = 0$

When $r = 2$: MEAN SQUARE CONVERGENCE $X_n \xrightarrow{2} X$

$$\lim_{n\to\infty} E\left(|X_n - X|^2\right) = 0$$

4. Let $F_n(x)$ and $F(x)$ the c.d.f. functions of X_n and X respectively.

X_n converges in distribution to X : $X_n \xrightarrow{d} X$

iff

$\lim\limits_{n\to\infty} F_n(x) = F(x)$ at all pts where $F(x)$ is continuous.

5 $X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

6. $X_n \xrightarrow{2} X \Rightarrow X_n \xrightarrow{P} X$

RANDOM SERIES

X_t, $t \in \mathbb{N}$ a stochastic process with real values and $\sum_{t=1}^{\infty} X_t$ a series

$\sum_{t=1}^{\infty} X_t$ converges iff $\sum_{t=1}^{N} X_t \xrightarrow[N\to\infty]{} Y$

- convergence can be in probability, in mean square or almost surely (a-s)

$\sum_{t=1}^{\infty} X_t$ converges in absolute values (a-s, p, 2) iff

$\sum_{t=1}^{\infty} |X_t|$ converges (a.s., p, 2) if $\sum_{t=1}^{\infty} |X_t|$ converges a.s. we write: $\sum_{t=1}^{\infty} |X_t| < \infty$ a.s

- $\sum_{t=1}^{\infty} |X_t| < \infty$ a.s $\Rightarrow \sum_{t=1}^{\infty} X_t$ converges a.s.

- $\sum_{t=1}^{\infty} X_t$ converges in absolute values in probability

  $\Longleftrightarrow \sum_{t=1}^{\infty} X_t$ converges in absolute values a.s.

- $\sum_{t=1}^{\infty} E|X_t| < \infty \Rightarrow \sum_{t=1}^{\infty} |X_t| < \infty$ a.s. and

  $$E\left[\sum_{t=1}^{\infty} |X_t|\right] = \sum_{t=1}^{\infty} E|X_t|$$

- if $E(X_t) = 0$ $\forall t$: $X_t \in L_2$

  $\sum_{t=1}^{\infty} \sigma_t < \infty \Rightarrow \sum_{t=1}^{\infty} |X_t|$ and $\sum_{t=1}^{\infty} X_t$ converges a.s & in 2

- $\sum_{t=1}^{\infty} E|X_t| < \infty \Rightarrow E\left[\sum_{t=1}^{\infty} X_t\right] = \sum_{t=1}^{\infty} E(X_t)$