Note that if

$$A(L) = 1 + a_1 L + a_2 L^2 \text{, then:} \quad \begin{array}{l} a_1 + a_2 > -1 \\ a_1 - a_2 < 1 \\ a_2 < 1 \end{array}$$

IN GENERAL

CONSIDER AN AR(p) PROCESS

$$A(L) X_t = \varepsilon_t$$

$$A(L) = 1 + a_1 L + a_2 L^2 + \dots + a_p L^p$$

THE AR(p) IS STATIONARY IF THE ROOTS OF $A(L) = 0$ ALL LIE OUTSIDE THE UNIT CIRCLE.

We can write:

$$A(L) = \prod_{i=1}^{p} (1 - \mu_i L)$$

↑ roots of the reciprocal

if $A(L) = 1 + a_1 L + a_2 L^2$

$\underline{\mu}$'s are roots of

$A(z) = z^2 + a_1 z + a_2$

Variance and Autocovariance Function of AR(2)

let's

$$\tilde{X}_t = \bar{\mu} + \sum_{j=1}^{p} a_j \tilde{X}_{t-j} + \varepsilon_t$$

ASSUME

1. $\tilde{X}_t$ IS STATIONARY WITH $\sum_{j=1}^{p} a_j \neq 1$

2. $E(\tilde{X}_{t-j}\, \varepsilon_t) = 0, \quad \forall j \geq 0$

BY STATIONARITY:

$$E(\tilde{X}_t) = \mu \qquad \forall t \Rightarrow$$

$$\mu = \bar{\mu} + \sum_{j=1}^{p} a_j \bar{\mu}$$

$$E(\tilde{X}_t) = \mu = \frac{\bar{\mu}}{1 - \sum_{j=1}^{p} a_j}$$

the demeaned process: X_t

$$X_t = \sum_{j=1}^{p} a_j X_{t-j} + \varepsilon_t$$

$$X_{t+K} = \sum_{j=1}^{p} a_j X_{t+K-j} + \varepsilon_{t+K} \qquad / \cdot X_t$$

$$E(X_{t+k}X_t) = \sum_{j=1}^{p} a_j E(X_{t+k-j} X_t) + E(\varepsilon_{t+k} X_t)$$

$$\gamma(k) = \sum_{j=1}^{p} a_j \gamma(k-j) + E(\varepsilon_{t+k} X_t)$$

where $E(\varepsilon_{t+k} X_t) = \sigma_\varepsilon^2$, if $k=0$

$= 0$, if $k \geq 1$

$$\rho(k) = \sum_{j=1}^{p} a_j \rho(k-j) \quad , \quad k \geq 1$$

WE CALL THESE FORMULAS: YULE-WALKER EQUATIONS.

IF WE KNOW $\rho(1), \dots, \rho(p)$ WE CAN COMPUTE $\rho(k)$ FOR $k \geq p+1$.

THE AUTOCORRELATIONS OF AR(p) ARE GRADUALLY EXPONENTIALLY DECAYING WITH k.

YULE-WALKER:

$$A(L)\, \rho(k) = 0 \qquad \forall\, k \geq 1$$

WHERE $L^j \rho(k) = \rho(k-j)$.

TO OBTAIN $\rho(1), \dots, \rho(p)$ WE SOLVE:

$$\rho(1) = a_1 + a_2\,\rho(1) + \cdots + a_p\,\rho(p-1)$$

$$\rho(2) = a_1\,\rho(1) + a_2 + \cdots + a_p\,\rho(p-2)$$

$$\vdots$$

$$\rho(p) = a_1\,\rho(p-1) + a_2\,\rho(p-2) + \cdots + a_p$$

WITH: $\rho(-j) = \rho(j)$

- AUTOCOV. OF ORDER $K > p$ ARE OBTAINED FROM:

$$\rho(K) = \sum_{j=1}^{p} a_j\,\rho(K-j)\,, \quad K \geq p+1$$

TO COMPUTE $\gamma(0) = var(X_t)$ WE SOLVE:

$$\gamma(0) = \sum_{j=1}^{p} a_j\,\gamma(-j) + E(\varepsilon_t X_t)$$

$$\gamma(0) = \sum_{j=1}^{p} a_j\,\gamma(j) + \sigma_\varepsilon^2$$

SINCE $\gamma(j) = \rho(j)\,\gamma(0) \Rightarrow$

$$\gamma(0)\left[1 - \sum_{j=1}^{p} a_j\,\rho(j)\right] = \sigma_\varepsilon^2$$

$$\gamma(0) = \frac{\sigma_\varepsilon^2}{1 - \sum_{j=1}^{p} a_j\,\rho(j)}$$

1). AR(1):

$$X_t = a_1 X_{t-1} + \varepsilon_t$$

$$\rho(1) = a_1$$

$$\rho(K) = a_1 \rho(K-1), \quad K \geq 1$$

$$\Rightarrow \quad \rho(2) = a_1 \rho(1) = a_1^2$$

$$\rho(K) = a_1^K, \quad K \geq 1$$

$$\gamma(0) = \mathrm{var}(X_t) = \frac{\sigma_\varepsilon^2}{1-a_1^2}$$ COMPARE WITH...

2) AR(2)

$$X_t = a_1 X_{t-1} + a_2 X_{t-2} + \varepsilon_t$$

$$\rho(1) = a_1 + a_2 \rho(2)$$

$$\rho(2) = a_1 \rho(1) + a_2 \quad \Rightarrow \quad \rho(1) = \frac{a_1}{1-a_2}$$

$$\rho(2) = \frac{a_1^2}{1-a_2} + a_2 = \frac{a_1^2 + a_2(1-a_2)}{1-a_2}$$

$$\gamma(0) = \frac{(1-a_2)\,\sigma_\varepsilon^2}{(1+a_2)(1-a_1-a_2)(1+a_1-a_2)}$$

FOR $K > p = 2$, i.e. FOR $K \geq 3$

$$\rho(K) = a_1 \rho(K-1) + a_2 \rho(K-2)$$

HOMOGENOUS, 2nd ORDER DIFF EQ: $\rho(K) = A_1 \mu_1^K + A_2 \mu_2^K$,

μ_1, μ_2 are roots of $A(z) = z^2 + a_1 z + a_2$.

PARTIAL AUTOCORRELATIONS

LINEAR FORECASTS:

CONSIDER A STATIONARY PROCESS X_t, FOR WHICH THE AUTOCORRELATION MATRICES $R(m)$ ARE ALL NONSINGULAR FOR INTEGER m.

WE ARE INTERESTED IN THE BEST LINEAR PREDICTION OF X_t BASED ON THE K PREVIOUS VALUES: $X_{t-1}, X_{t-2}, \dots X_{t-K}$

ASSUME X_t IS DEMEANED.

THE PREDICTION:

$$E(X_t \mid X_{t-1}, \dots, X_{t-K}) = a_1(K)\, X_{t-1} + \dots + a_K(K)\, X_{t-K}$$

WHERE THE VECTOR $a(K) = (a_1(K), \dots a_K(K))'$ OF THE REGRESSION COEFF IS:

$$a(K) = R(K)^{-1} \begin{bmatrix} \rho(1) \\ \vdots \\ \rho(K) \end{bmatrix}$$

FOR AR(p):

$$\rho(K) = \sum_{j=1}^{p} a_j\, \rho(K-j) \qquad , K = \dots$$

GIVEN THAT:

$$\rho(0) = 1 \quad \text{AND} \quad \rho(-K) = \rho(K)$$

AND a_j DENOTE HERE THE AUTOREGRESSIVE COEFF.

$$\begin{bmatrix} 1 & \rho(1) & \rho(2) & \cdots & \rho(p-1) \\ \rho(1) & 1 & \rho(1) & \cdots & \rho(p-2) \\ \vdots & & & & \\ \rho(p-1) & \rho(p-2) & \rho(p-3) & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(p) \end{bmatrix}$$

$$R \cdot a = \rho$$

$$a = R^{-1} \rho$$

INTRODUCING DIMENSIONS, WE HAVE SEQUENCE OF EQUATIONS:

$$R_K a_K = \rho_K \qquad k = 1, 2, 3 \ldots$$

WHERE

$$a_K = [a_{K1}, a_{K2}, \ldots a_{KK}]'$$

$$a_K = R_K^{-1} \rho_K.$$

THE COEF a_{KK} IS EQUAL TO THE COEFF OF CORRELATION

BTW:
$$X_t - E(X_t \mid X_{t-1}, \ldots, X_{t-K+1})$$

AND
$$X_{t-K} - E(X_{t-K} \mid X_{t-1}, \ldots, X_{t-K+})$$

IT MEASURES THE LINEAR LINK BTW X_t AND X_{t-K} ONCE THE INFLUENCE OF INTERVENING VARIABLES $X_{t-1}, \ldots, X_{t-K+1}$ HAS BEEN REMOVED

p. 130 G.M.

FOR AN AR(p) WE HAVE

$$a_{KK} = 0 \qquad \forall K \geq p+1$$!

SOME a_{KK}:

$$a_{11} = \rho_1$$

$$a_{22} = \frac{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & \rho_2 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{vmatrix}} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

$$a_{33} = \frac{\begin{vmatrix} 1 & \rho_1 & \rho_1 \\ \rho_1 & 1 & \rho_2 \\ \rho_2 & \rho_1 & \rho_3 \end{vmatrix}}{\begin{vmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_1 \\ \rho_2 & \rho_1 & 1 \end{vmatrix}} =$$

DURBIN - LEVINSON

$$a_{K+1,K+1} = \frac{\rho(K+1) - \sum_{j=1}^{K} a_{Kj}\,\rho(K+1-j)}{1 - \sum_{j=1}^{K} a_{Kj}\,\rho(j)}$$

$$a_{K+1,j} = a_{Kj} - a_{K+1,K+1}\, a_{K,K-j+1} \quad , \quad j = 1,2 \dots K$$