SEASONALITY

1. Seasonal MA(q)

$$X_t = B_q(L^s)\,u_t$$

$$B_q = 1 - b_1 L^s - b_2 L^{2s} - \dots - b_q L^{qs}$$

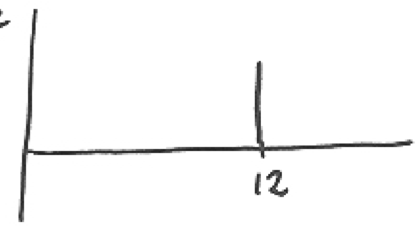
where $s = 4, 12, 52$ for example.

$$X_t = u_t - b_1 u_{t-s} - b_2 u_{t-2s} - \dots - b_q u_{t-qs}$$

ex:

$MA(1)_{12}$: $X_t = u_t - b_1 u_{t-12}$

ACF

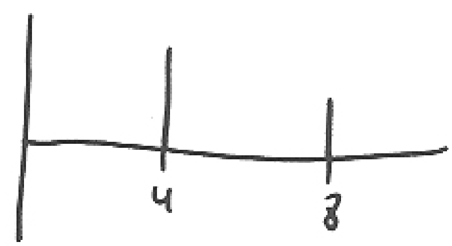


$MA(2)_4$; $s = 4, q = 2$

$$X_t = [1 - b_1 L^4 - b_2 L^8]\,u_t$$

$$= u_t - b_1 u_{t-4} - b_2 u_{t-8}$$

ACF



AUTOCORREL AT :

$$\rho_s,\ \rho_{2s} \dots,\ \rho_{qs} \neq 0$$

AND CUT OFF AT qs !

2. Seasonal AR(p)

$$A_p(L^s)X_t = u_t$$

$$A_p(L^s) = 1 - a_1 L^s - a_2 L^{2s} - \dots - a_p L^{ps}$$

ex

AR(1)$_s$ $$X_t = a_1 X_{t-s} + u_t$$

$$(1 - a_1 L^s) X_t = u_t$$

$$X_t = \frac{u_t}{1 - a_1 L^s} = u_t + a_1 u_{t-s} + a_1^2 u_{t-2s} \ldots$$

IN A MA(∞) SEASONAL REPRESENTATION

$$\gamma(0) = \frac{\sigma^2}{1 - a_1^2}$$

$$\gamma_{js} = a_1^j \gamma_0 \qquad j = 0, 1, 2$$

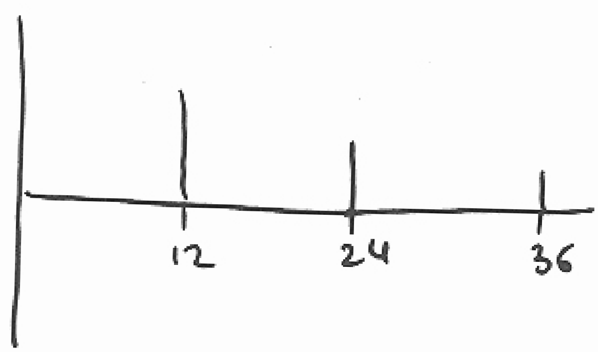
$$\rho_{js} = a_1^j \quad , \qquad j = 0, 1, 2 \ldots$$

$$\rho_K = 0 \quad , \qquad \text{for } K \neq js$$

ALL AUTOCORREL EXCEPT js (MULTIPLE OF S) ARE 0

$$X_t = 0.8 X_{t-12} + u_t$$

ACF



no ? ✓ ? ✓ ✓

AR(1) AR(2) ARMA(1,1) ARMA(1;1,4) ARMA(1,2)

Table 2.4: **Estimates of the WPI (Logarithmic First Differences)**

	$p=1$ $q=0$	$p=2$ $q=0$	$p=1$ $q=1$	$p=1$ $q=1,4$	$p=1$ $q=2$
a_0	0.011 (4.14)	0.011 (3.31)	0.012 (2.63)	0.011 (2.76)	0.012 (2.62)
a_1	0.618 (8.54)	0.456 (5.11)	0.887 (14.9)	0.791 (9.21)	0.887 (13.2)
a_2		0.258 (2.89)			
β_1			−0.484 (−4.22)	−0.409 (−3.62)	−0.483 (−4.19)
β_2					−0.002 (−0.019)
β_4				0.315 (3.36)	
SSR	0.0156	0.0145	0.0141	0.0134	0.0141
AIC	−503.3	−506.1	−513.1	−518.2	−511.1
SBC	−497.7	−497.7	−504.7	−507.0	−499.9
$Q(12)$	23.6 (0.008)	11.7 (0.302)	11.7 (0.301)	4.8 (0.898)	11.7 (0.301)
$Q(24)$	28.6 (0.157)	15.6 (0.833)	15.4 (0.842)	9.3 (0.991)	15.3 (0.841)
$Q(30)$	40.1 (0.082)	22.8 (0.742)	22.7 (0.749)	14.8 (0.972)	22.6 (0.749)

Notes: Each coefficient is reported with the associated t-statistic for the null hypothesis that the estimated value is equal to zero.

SSR is the sum of squared residuals.

$Q(n)$ reports the Ljung–Box Q-statistic for the autocorrelations of the n residuals of the estimated model. With 122 observations, $T/4$ is approximately equal to 30. Significance levels are in parentheses.

[illegible] Figure 2.5 suggests that [illegible] seems [illegible] the sample data.

[illegible] such that [illegible] the PACF [illegible]

[illegible]

Figure 2.5 U.S. wholesale price index (1985 = 100).

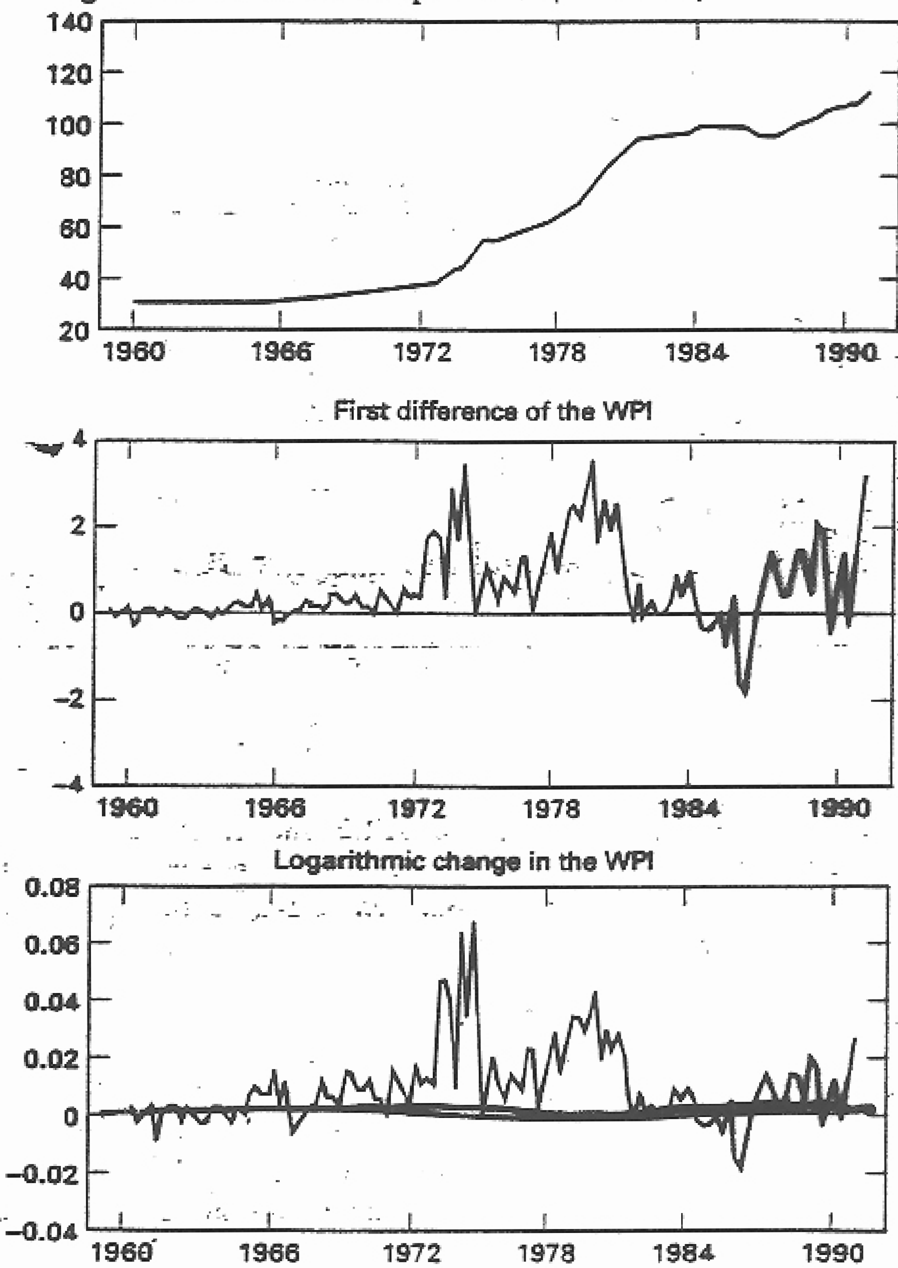


Figure 2.6 ACF and PACF for the logarithmic change in the WPI.

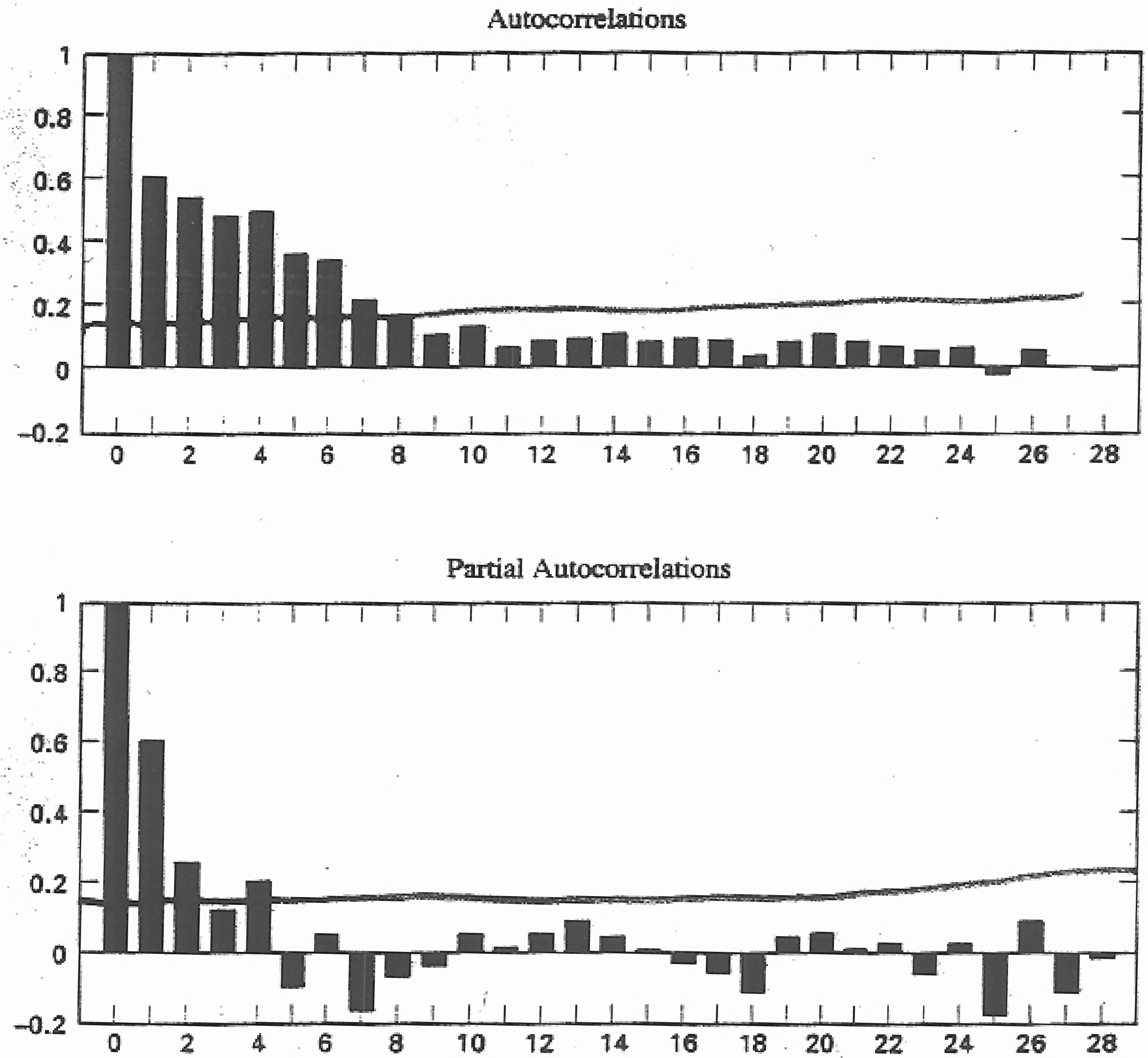


Figure 2.7 Tourism in Spain.

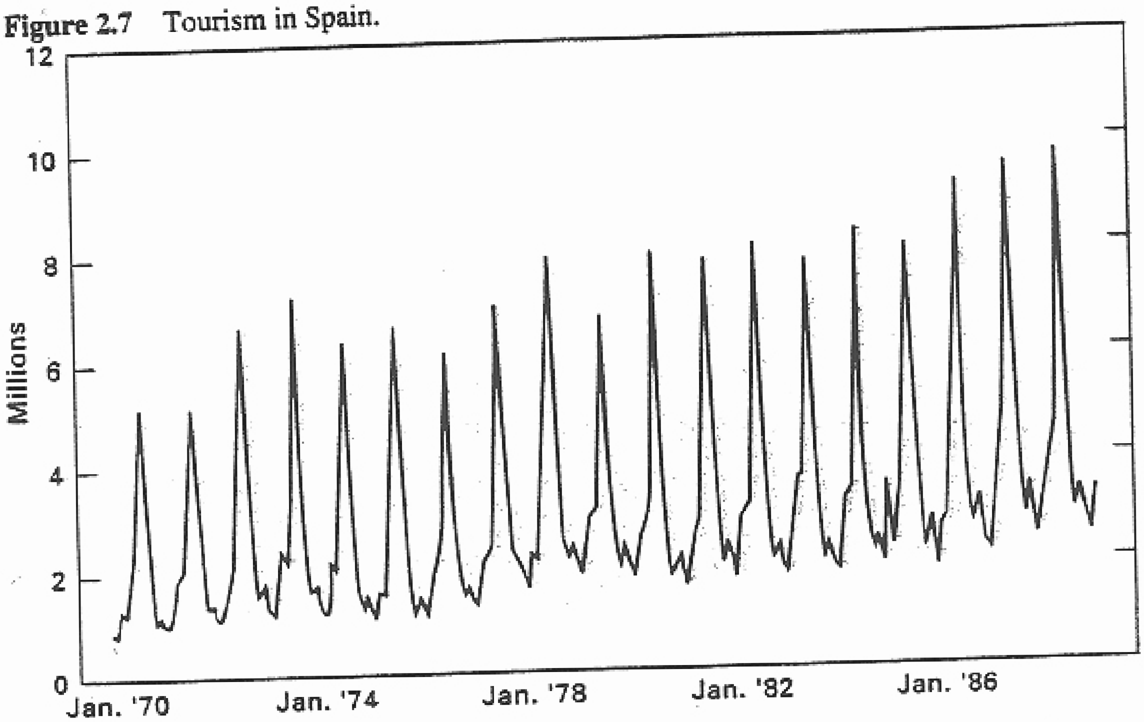


Figure 2.8 Correlogram of tourism in Spain.

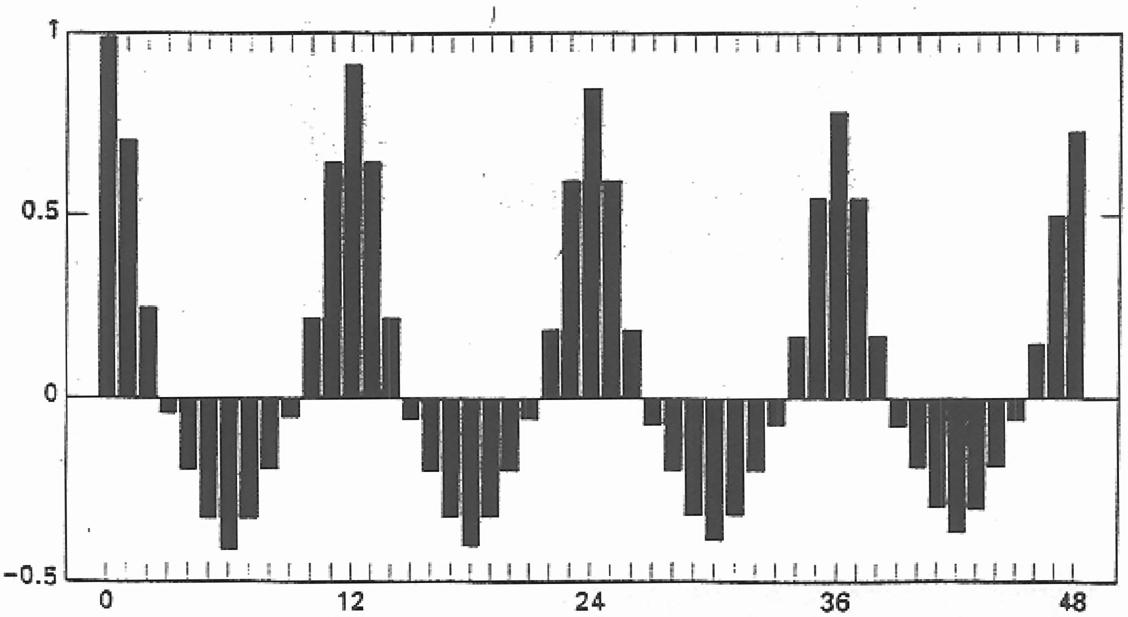


Figure 2.9 First and twelfth differences.

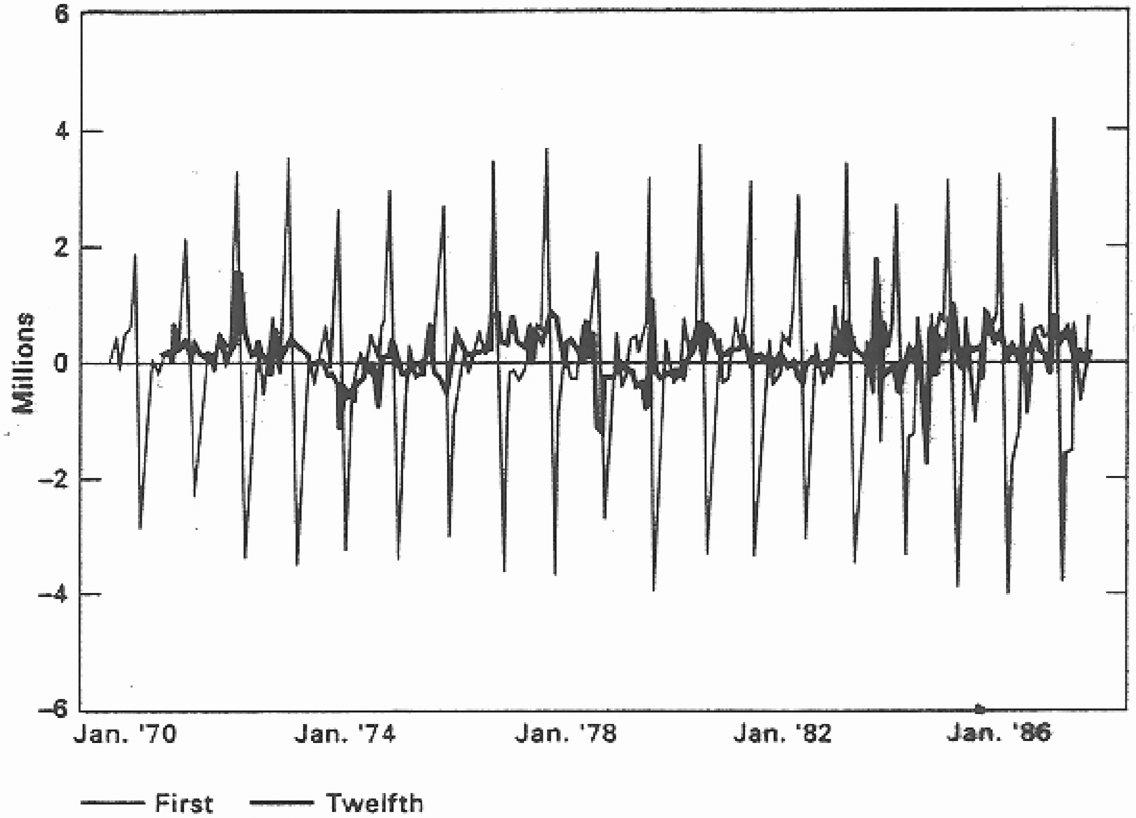


Table 2.5: **Three Models of Spanish Tourism**

	Model 1[1]	Model 2	Model 3
a_{12}	−0.408 (−6.54)		
β_1	−0.738 (−15.56)	0.740 (−16.14)	−0.640 (−14.75)
β_{12}		−0.671 (−13.02)	−0.306 (−7.00)
SSR	2.823	2.608	3.367
AIC	217.8	212.98	268.70
SBC	224.5	219.75	275.47
$Q(12)$	8.59 (0.571)	4.38 (0.928)	25.54 (0.004)
$Q(24)$	41.11 (0.007)	15.71 (0.830)	66.58 (0.000)
$Q(48)$	67.91 (0.019)	37.61 (0.806)	99.31 (0.000)

Clearly, there is no difference between an additive seasonality and multiplicative seasonality when all other autoregressive coefficients are zero.

3. ARIMA SEASONAL

ARMA: $A_p(L^s)X_t = B_q(L^s)u_t + \tilde{\mu}$

$$\rho_K = 0 \quad \text{for } K \neq js$$

ALL OTHER AUTOCORREL ARE ZERO

ARIMA:

$$A_p(L^s)(1-L^s)X_t = B_q(L^s)u_t + \tilde{\mu}$$

HAS TO BE DIFFERENTIATED: $X_t - X_{t-s}$ TO REMOVE THE SEASONAL UNIT ROOT

MULTIPLICATIVE SEASONALITY

$$A_p(L)(1-L)^d \, \Phi_{P}(L^s)(1-L^s)^D X_t =$$

$$= B_q(L)\,\Theta_Q(L^s)u_t + \tilde{\mu}$$

ARIMA(p,d,q) × ARMA$(s)(P,D,Q)$

FORECASTING IN ARIMA

$$(1-L)^d A_p(L)\, X_t = B_q(L)u_t + \tilde{\mu}$$

$$u_t \sim WN.(0,\sigma^2)$$

$$A(L)\,X_t = B(L)u_t + \tilde{\mu}$$

$A(L) = (1-L)^d \, A_p(L)$

$$\left[1 - a_1L - \cdots - a_{p+d}L^{p+d}\right]X_t = \left[1 - b_1L - \cdots - b_qL^q\right]u_t + \tilde{\mu}$$

$$X_t = \sum_{i=1}^{p+d} a_i X_{t-i} - \sum_{j=1}^{q} b_j u_{t-j} + u_t + \bar{\mu}$$

WE NEED:

$$E_t X_{t+1} = \sum_{i=1}^{p+d} a_i E_t X_{t+1-i} - \sum_{j=1}^{q} b_j E_t u_{t+1-j} + \bar{\mu} + \underbrace{E_t u_{t+1}}_{=0}$$

$$= \sum_{i=1}^{p+d} a_i X_{t+1-i} - \sum_{j=1}^{q} u_{t+1-j} + \bar{\mu}$$

$$E_t X_{t+2} = a_1 E_t X_{t+1} + \sum_{i=2}^{p+d} a_i X_{t+2-i} - \sum_{j=2}^{q} b_j u_{t+2-j} + \bar{\mu} + \underbrace{E_t u_{t+2}}_{=0}$$

$$E_t X_{t+l} = \sum_{i=1}^{l-1} a_i E_t X_{t+l-i} + \sum_{i=l}^{p+d} X_{t+l-i} - \sum_{j=l}^{q} b_j u_{t+l-j} + \bar{\mu}$$

- $E_t X_{t+l} = X_{t+l}$, for $l \leq 0$

- $E_t u_{t+l} = \begin{cases} 0 , & l \geq 1 \\ u_{t+l} , & l \leq 0 \end{cases}$

$$E_t X_{t+l} = \sum_{i=1}^{p+d} a_i E_t X_{t+l-i} + E_t u_{t+l} - \sum_{j=1}^{q} b_j E_t u_{t+l-j} + \bar{\mu}$$

$$A(L) E_t X_{t+l} = B(L) E_t u_{t+l} + \hat{\mu}$$

THE FORECAST IS ARMA TOO!

$$- \quad A(L) X_{t+l} = B(L) u_{t+l} + \bar{\mu}$$

$$A(L)(X_{t+l} - E_t X_{t+l}) = B(L)(u_{t+l} - E_t u_{t+l})$$

$$A(L)\left(x_{t+l} - E_t X_{t+l}\right) = B(L)\left(u_{t+l} - E_t u_{t+l}\right)$$

CONSIDER

$$\Psi(L) = \sum_{i=0}^{\infty} \psi_j L^j \quad ; \quad A(L)\,\Psi(L) = B(L)$$

CALL $e_t(l) = u_{t+l} - E_t u_{t+l}$ THE FORECAST ERROR OF $x_{t+l} - E_t x_{t+l}$

$$A(L)\, e_t(l) = A(L)\,\Psi(L)\left(u_{t+l} - E_t u_{t+l}\right)$$

NOTE THAT:

$$e_t(l) = 0 \quad \text{if } l \leq 0$$

$$u_{t+l} - E_t u_{t+l} \begin{cases} = 0 & \text{if } l \leq 0 \\ = u_{t+l} \; ; & l > 1 \end{cases}$$

$$e_t(l) = \Psi(L)\left(u_{t+l} - E_t u_{t+l}\right)$$

$$e_t(l) = u_{t+l} + \psi_1 u_{t+l-1} + \cdots + \psi_{l-1} u_{t+1}$$

- $$e_t(l) = \sum_{i=0}^{l-1} \psi_i u_{t+l-i} \quad ; \quad \psi_0 = 1$$

PREDICTION ERROR l PERIODS AHEAD IS $MA(l-1)$

PROPERTIES:

- UNBIASED: $E_t\left[e_t(l)\right] = 0$
- $\text{var}\; e_t(l) = \sigma^2\left[1 + \sum_{j=1}^{l-1} \psi_j^2\right]$

NOTE! IN PRESENCE OF UNIT ROOT: $var(e_t(l)) \longrightarrow \infty$ WITH l

($\sum \psi_j^2$ IS NOT FINITE)

IN ANY OTHER CASE $var(e_t(l)) \uparrow$ WITH l.

STRUCTURE OF $e_t(l)$

$e_t(1)$

for $l \geq 2$ THE FORECAST ERROR FOLLOWS $MA(l-1)$

$$E\left[e_t(l)\, e_{t-j}(l)\right] = \begin{cases} \sigma^2 \sum_{i=j}^{l-1} \psi_i \psi_{i-j} & 0 \leq j \leq l-1 \\ 0 & |j| \geq l \end{cases}$$

$$E\left[e_t(l)\, e_t(l+j)\right] = \left(\sum_{i=0}^{l-1} \psi_i \psi_{i+j}\right)\sigma^2$$

CONFIDENCE INTERVALS

ASSUME $u_t \overset{i.i.d.}{\sim} N(0, \sigma^2)$

$e_t(l) \sim N(0, v_l^2)$, where $v_l^2 = \sigma^2\left[1 + \sum_{j=1}^{l-1} \psi_j^2\right]$

$$P\left[E_t X_{t+l} - c_{\alpha/2} v_l \leq X_{t+l} \leq E_t X_{t+l} + c_{\alpha/2} v_l\right] = 1-\alpha$$

FORECAST FUNCTION

$$\hat{X}_t(l) = E_t X_{t+l}, \quad l = 1,2,3$$

$$A(L)\, \hat{X}_t(l) = B(L) \quad u_{t+l} + \bar{\mu}$$

$\downarrow$

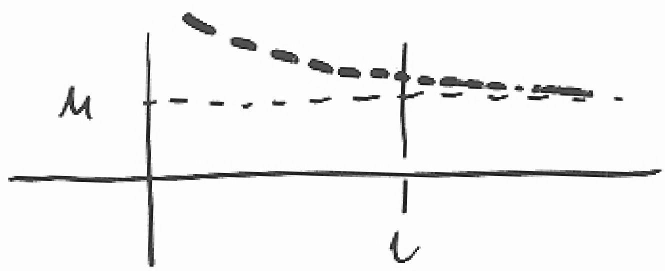
$$A_p(L)(1-L)^d\, \hat{X}_t(l) = B(L) \quad u_{t+l} + \bar{\mu}$$

- if $d=0$, l large.

$$A_p(L)\ \hat{X}_t(l) = \bar{\mu}$$

$$\hat{X}_t(l) = \frac{\bar{\mu}}{A_p(L)} = \frac{\bar{\mu}}{1-a_1-\dots-a_p} = \mu$$

IF WE HAVE AN ARMA(p,q) STATIONARY PROCESS, THE FORECAST APPROACHES THE UNCONDITIONAL MEAN WHEN $l\uparrow$



- if $d=1$ NONSTATIONARY ARIMA(p,d,q)

l large

$$A_p(L)\ (1-L)\ \hat{X}_t(l) = \bar{\mu}$$

$$(1-L)\ \hat{X}_t(l) = \frac{\bar{\mu}}{A_p(L)} = \mu$$

$$\hat{X}_t(l) - \hat{X}_t(l-1) = \mu \Rightarrow \hat{X}_t(l) \simeq \mu + \mu l$$