# 1. INTRODUCTION

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# 1. Introduction

Originally focus on conditional mean models but as economic theory started to consider risk behavior.

→ higher order moments (non-linearities)
→ dynamics of uncertainty

We will cover class of models called ARCH-type models.

<u>Original paper</u> :

Engle, R.F. (1982), "Autoregressive conditional heteroscedasticity with estimators of the variance of U.K. inflation", *Econometrica* 50, 987-1007.

<u>Survey papers</u> :

Bollerslev, T., R.Y. Chou and K.F. Kroner (1992), "ARCH Modeling in Finance : A Review of the Theory and Empirical Evidence", *Journal of Econometrics* 52, 5-59.
(Lists 200 + applications - Finance)

Bollerslev, T., R.F. Engle and D.B. Nelson (1994), "ARCH Models", *Handbook of Econometrics*, vol. 4 (forthcoming).

## 1.1 DEFINITIONS

Let $\{\varepsilon_t(\theta)\}$ denote a discrete time stochastic process with cond. mean and variance functions parameterized by finite dimensional vector $\theta \in \Theta \subseteq R^n$ and $\theta_0$ denotes the true value.

For the moment : $\varepsilon_t(\theta)$ is scalar.

The process $\{\varepsilon_t\,(\theta_0)\}$ is ARCH if

$$E_{t-1}\,(\varepsilon_t\,(\theta_0)) = 0 \qquad\qquad t = 1,2,\ldots$$

$$\sigma_t^2(\theta_0) \equiv Var_{t-1}\,(\varepsilon_t(\theta_0)) = E_{t-1}\left(\varepsilon_t^2(\theta_0)\right)$$

depends non-trivially on the <u>sigma-field</u> generated by

$$\left\{\varepsilon_{t-1}(\theta_0), \varepsilon_{t-2}(\theta_0), \ldots\right\}$$

Note where there is action in the mean

$$\mu_t\,(\theta_0) = E_{t-1}\,(y_t)$$

Then, let $\varepsilon_t(\theta_0) \equiv y_t - \mu_t\,(\theta_0)$

$\Rightarrow$ cond. var. of $\varepsilon_t\,(\theta_0) \equiv$ cond. var. of $y_t\,(\theta_0)$

From the analysis so far, we have that :

$$z_t(\theta_0) \equiv \varepsilon_t(\theta_0)\left(\sigma_t^2(\theta_0)\right)^{-1/2} \quad \Rightarrow \quad \text{standardized process}$$

$\Rightarrow$ $z_t(\theta_0)$ has cond. mean zero
cond. var. = 1

This is the basis of estimation of ARCH type models

**Note :** If cond. distr. of $z_t$ has also time invariant fourth moments, then from Jensen's inequality :

$$E\left(\varepsilon_t^4\right) = E\left(z_t^4\sigma_t^4\right) = E\left(z_t^4\right) E\left(\sigma_t^4\right) \geq$$

$$E\left(z_t^4\right) E\left(\sigma_t^2\right)^2 = E\left(z_t^4\right)\left(E\varepsilon_t^2\right)^2$$

If $z_t \sim N(0,1) \Rightarrow \varepsilon_t$ is leptokurtic.

Fig 1.5 Monthly CRSP Equity Returns 1926/1 to 1989/12

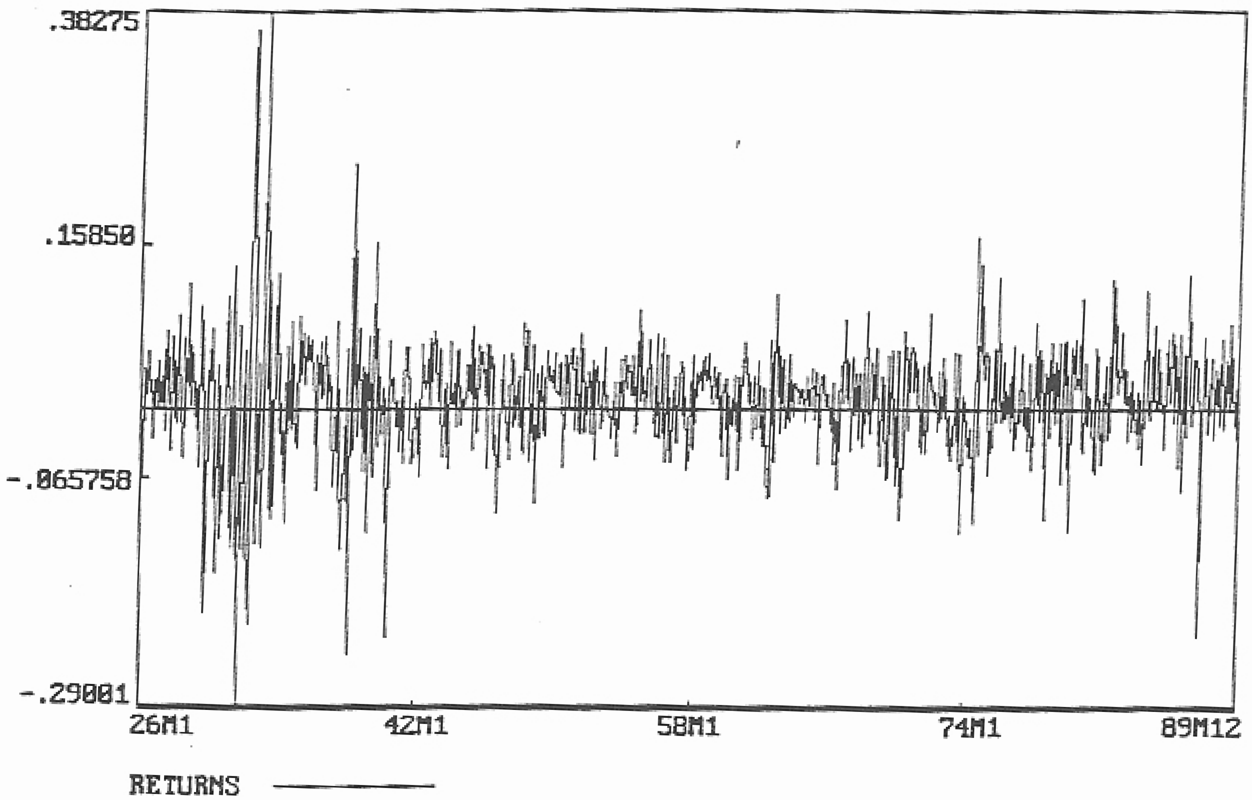
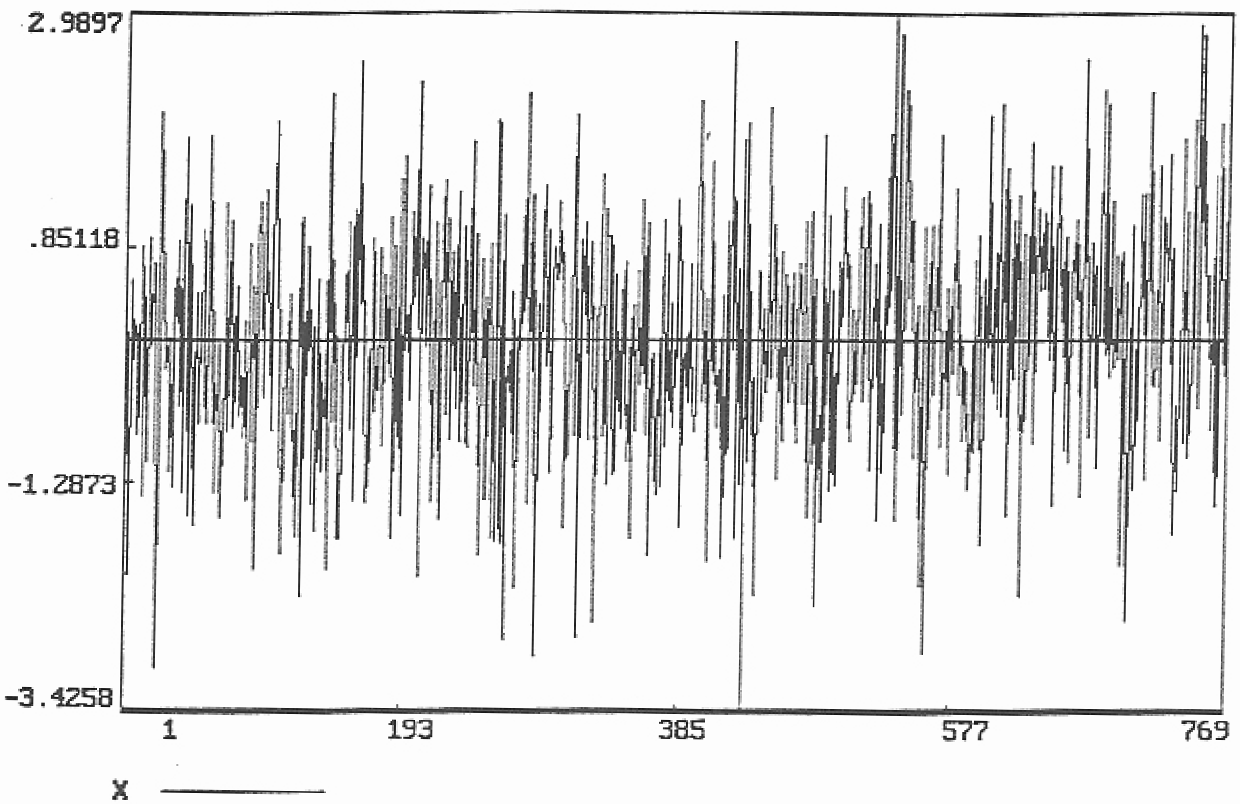


Fig 1.1 Time Series of Gaussian Noise



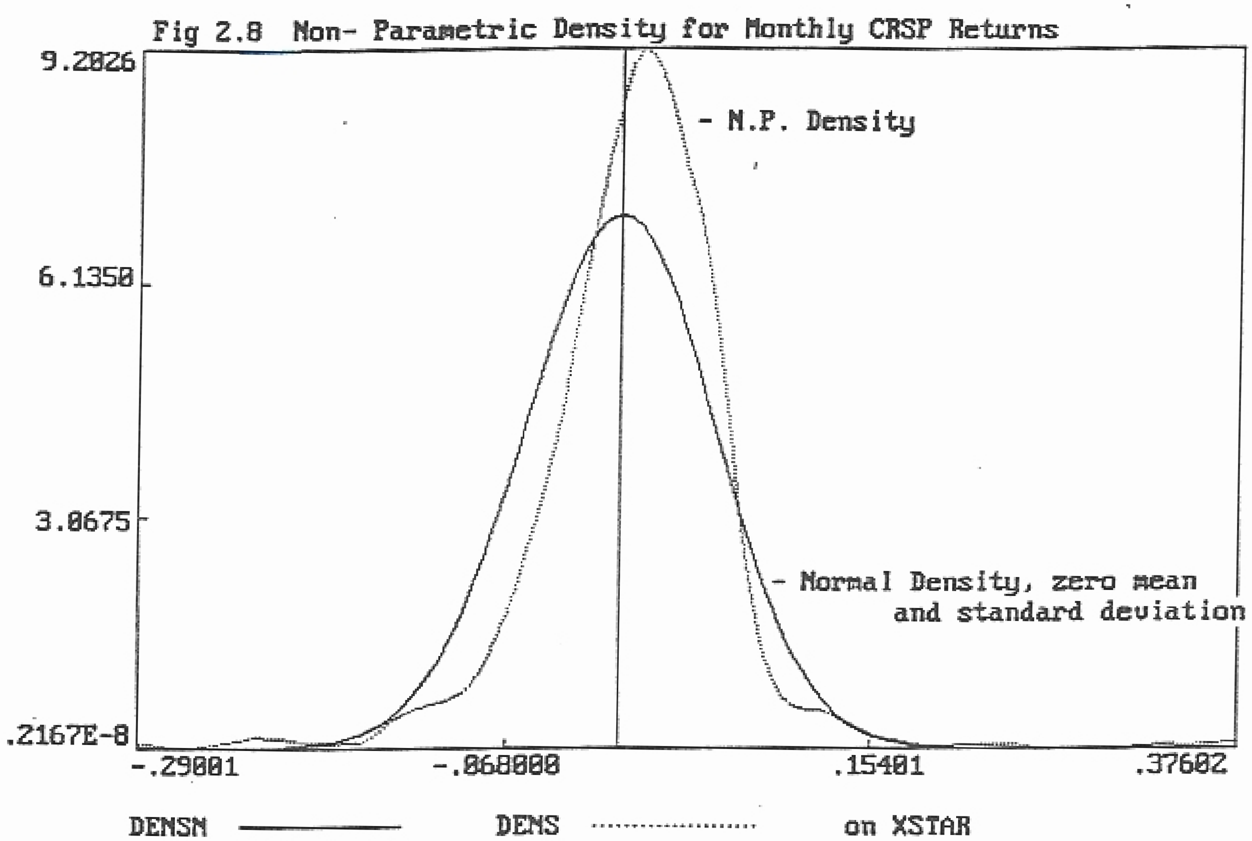


Fig 2.8 Non- Parametric Density for Monthly CRSP Returns
9.2026
6.1350
3.0675
.2167E-8
- N.P. Density
- Normal Density, zero mean
and standard deviation
-.29001
-.068000
.15401
.37602
DENSN
DENS
on XSTAR

Fig 2.10 Non-parametric Density, change in log $US/SF

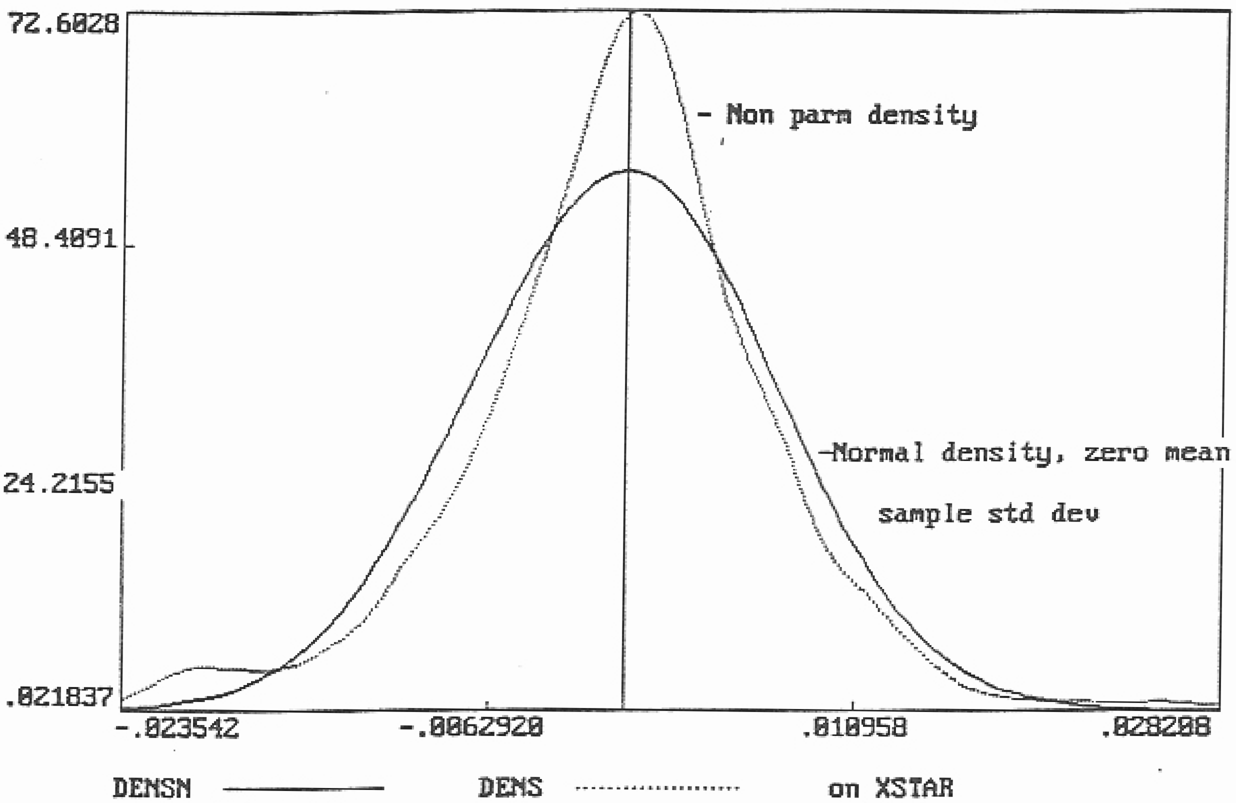
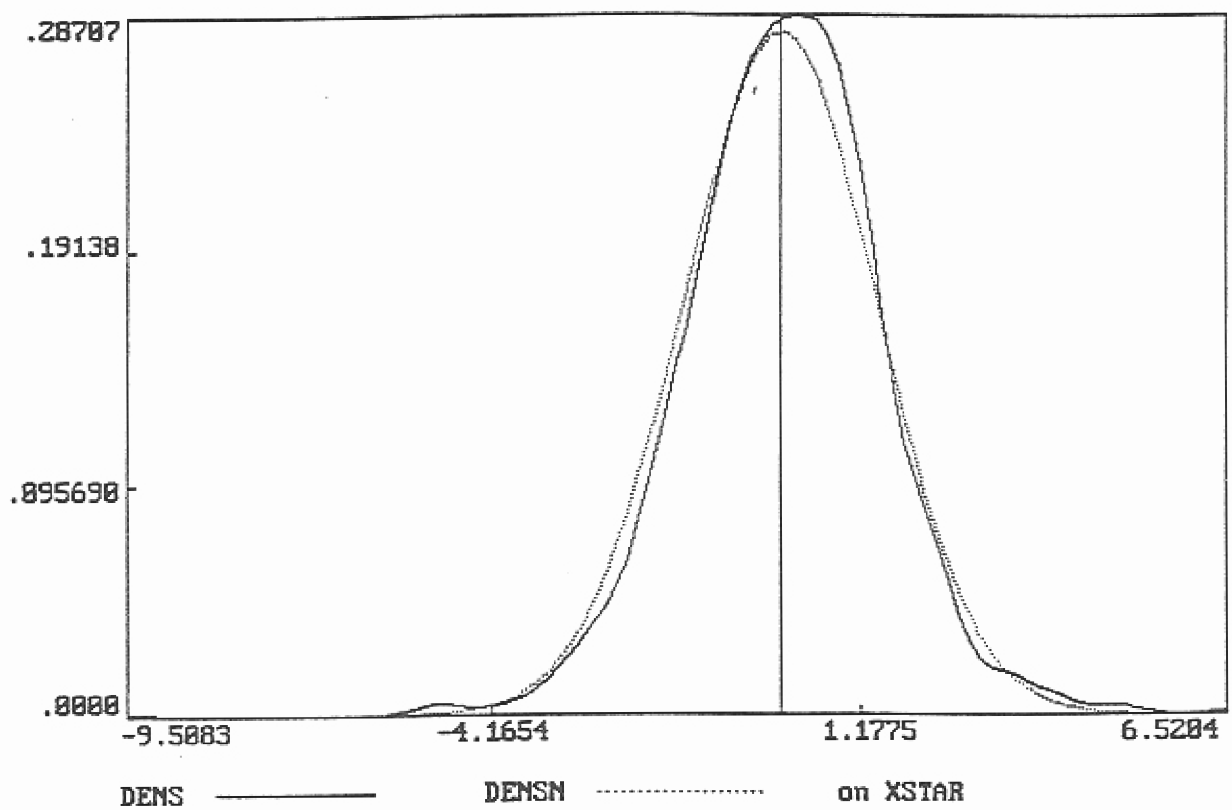


Fig 2.11 Estimated Density for an ARCH(1) Process



## 1.2 EMPIRICAL REGULARITIES OF ASSET RETURNS

Parametric specification of $\varepsilon_t(\theta_0)$ has many possibilities

$\Rightarrow$ to have any guidance, study stylized facts first

***(i) Thick tails***

Asset returns are leptokurtic

***(ii) Volatility Clustering***

$\Rightarrow$ look at any plot of financial time series, i.e. stocks, exchange rate, etc.

French, Schwert and Stambaugh (1987) Volatility during 30's > Volatility during 60's

**Note :** Thick tails and volatility clustering are related

### *(iii) Leverage effects*

Black (1976)

Leverage effect = tendency for changes in stock prices to be negatively correlated with changes in stock volatility.

Firm with debt and equity becomes highly leveraged when value of the firm falls.

### *(iv) Non-Trading Periods*

Fama (1965), French and Roll (1986).

Information accumulates more slowly when markets are closed.

### *(v) Forecastable Events*

Forecastable events of important info $\Rightarrow$ high ex ante volatility.

Indiv. firm's stock returns volatility around earnings announcements.

Also fixed income and foreign exchange volatility increase during periods of central bank trading.

Volatility at the open and close of market and middle of day.

### *(vi) Volatility and Serial Correlation*

Strong inverse relation between volatility and serial correlation for U.S. stock indices.

### *(vii) Co-movements in volatilities*

Commonality in volatility.

⇒ reason for factor ARCH

### *(viii) Macroeconomic Variables and Volatility*

Surprisingly weak link between macroeconomic incertainty and volatility.

Glosten, Jagannathan and Runkle (1993).

Strong positive relationship between stock return volatility and interest rates.

## 1.3 Univariate Parametric Models

### *(i) ARCH (q)*

$$\sigma_t^2 = \omega + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 \equiv \omega + \alpha(L)\varepsilon_{t-1}^2$$

Define $\nu_t \equiv \varepsilon_t^2 - \sigma_t^2$

$$\Rightarrow \quad \varepsilon_t^2 = \omega + \alpha\,(L)\varepsilon_{t-1}^2 + \nu_t$$

AR(q) model for the squared innovations $\varepsilon_t^2$

Unconditional variance

$$\mathrm{Var}\,(\varepsilon_t) \equiv \sigma^2 = \omega \,/\, (1-\alpha_1 - .. - \alpha_q)$$

$$(\alpha_1 + \ldots + \alpha_q) < 1$$

$$\omega \geq 0 \qquad \alpha_i \geq 0 \qquad\qquad i = 1,\ldots,q$$

ARCH produces fat tails in the unconditional distribution,

**YET NOT FAT ENOUGH.**

The ARCH(q) model may also be represented as a time varying parameter AR(q) model for $\varepsilon_t$

$\varepsilon_t = \omega + \alpha(L)\zeta_{t-1}\,\varepsilon_{t-1}$ $\{\zeta_t\}$ scaler i.i.d. stochastic process with mean zero and variance one.

See Tsay (1987) and Bera, Higgins and Lee (1992) for further discussion.

In practice ARCH requires long lags to accommodate data

$\Rightarrow$ GARCH(p,q)

$$\sigma_t^2 = \omega + \sum_{i=1,q} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1,p} \beta_j \sigma_{t-j}^2 \equiv \omega + \alpha(L)\varepsilon_{t-1}^2 + \beta(L)\sigma_{t-1}^2$$

Positivity conditions on $\sigma_t^2$

*a)* $\alpha(L)$ and $\beta(L)$ have no common roots

*b)* the roots of $\beta(x) = 1$ lie outside the unit circle

*c)* $\alpha(x)/(1-\beta(x))$ all coeff. in this series are non-negative

Alternative representation for GARCH(p,q) process

Let $\quad \nu_t = \varepsilon_t^2 - \sigma_t^2$ ,

then $\quad \varepsilon_t^2 = \omega + \sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i)\varepsilon_{t-i}^2 - \sum_{j=1}^{q} \beta_j \nu_{t-j} + \nu_t$

Like ARMA(max(p,q),q) process on $\varepsilon_t^2$

$\Rightarrow$ stationarity : roots of $\alpha(x) + \beta(x) = 1$ lie outside the unit circle

Yet in many empirical applications

$\alpha(1) + \beta(1) \approx 1$

$\Rightarrow$ IGARCH processes (see later on issues of unit roots and persistence)

***(ii) EGARCH***

GARCH captures successfully
- thick tailed returns
- volatility clustering

can be modified to allow for - nontrading day effects
- predictable information releases

But <u>fails</u> in leverage effects

EGARCH of Nelson (1991)

$\sigma_t^2$ depends on both the size and the sign of lagged residuals

$$\ln\left(\sigma_t^2\right) = \omega + \left(1 + \sum_{i=1}^{q} \alpha_i L^i\right)\left(1 - \sum_{j=1}^{p} \beta_j L^j\right)^{-1} \left(\theta z_{t-1} + \gamma\left[\,|z_{t-1}| - E\,|z_{t-1}|\,\right]\right)$$

$\ln\left(\sigma_1^2\right)$ follows ARMA(p,q) process. Yet, innovation process is constructed to produce asymmetric response function $z_t \sim$ i.i.d. (0,1)

$\omega$ can accommodate non trading day effects etc.

***(iii) Other univariate parameterizations***

* Quite many, yet too often ad hoc
(see Bollerslev, Engle and Nelson (1994) for details).

## 1.4 ARCH IN MEAN MODELS

Many theories in finance call for an implicit trade off between the expected returns and the variance or the covariance between returns

CAPM

excess return on all risky assets are proportional to the non-diversifiable risk as measured by the covaiances with the market portfolio

Intertemporal CAPM Merton (1973)

expected excess return on the market portfolio is linear in its conditional variance (with log utility for representative agent)

$\Rightarrow$ ARCH in mean or ARCH-M
Engle, Lilien and Robins (1987)

$$\mu_t(\theta) = g(\sigma_t^2(\theta), \theta)$$

$$\partial g / \partial \sigma^2 \neq 0$$

## 1.5 Nonparametric and Semiparametric Models

Overwhelming variety of parametric univariate ARCH models

⇒ perhaps nonparametric approach

Pagan and Schwert (1990) : Kernel and Fourier series

To fit models for the relation between $y_t^2$ on past $y_t$s

$$y_t^2 = f\ (y_{t-1},\ y_{t-2},..\ y_{t-p},\theta)\ *\ \eta_t$$

Because of the problems of high dimensionality cannot choose p very high

⇒ limited possibility to capture temporal dependence

Issue of distance measure in fit

→ least squares o.k. for homoscedasticity but heteroscedasticity

# An Empirical Example of the EGARCH Model

Data : VWR daily returns from CRSP July 1962 - Dec. 1987

Minus (monthly T-Bill returns)

Denote raw data by $R_t$

Cond. Mean : $R_t = a + bR_{t-1} + c\sigma_t^2 + \xi_t$

$$E_{t-1}\xi_t = 0 \text{ and } E_{t-1}(\xi_t)^2 = \sigma_t^2$$

Cond. Var. : $\ln(\sigma_t^2) = \alpha_t + \dfrac{(1 + \psi_2 L + \ldots + \psi_q L^q)}{(1 - \Delta_1 L - \ldots - \Delta_p L^p)} g(z_{t-1})$

$\alpha_t = \alpha + \ln(1 + N_t\delta)$, $N_t$ : # nontrading days before day t

$$g(z_{t-1}) \equiv \theta Z_t + \lambda[|Z_t| - E|Z_t|]$$

$$z_t \sim \text{i.i.d.} \quad f(z) = \frac{\upsilon \exp[-(1/2)|z/\lambda|^{\upsilon}]}{\lambda 2^{(1+\upsilon^{-1})}\Gamma(1/\upsilon)}$$

$$\lambda \equiv [2^{(-2/\upsilon)}\Gamma(1/\upsilon)\Gamma(3/\upsilon)]^{1/2}$$

When $\upsilon = 2 \rightarrow$ Gaussian distr.

$\upsilon < 2 \rightarrow$ Thicker tails

$\upsilon > 2 \rightarrow$ Thinner tails

Useful expression for the likelihood :

$$Z_t = \sigma_t^{-1}\left(R_t - a - bR_{t-1} - c\sigma_t^2\right)$$

# 2. INFERENCE PROCEDURES

## 2.1 Testing for ARCH

## 2.2 MLE

## 2.3 Quasi - MLE

## 2.4 Specification Tests

## 2.1 Testing for ARCH

### *(i) Serial Correlation and LM tests*

Engle (1982) proposed a single test for ARCH

Under $\qquad H_0 \quad y_t = x_t\beta + \varepsilon_t$

$x_t$ is weakly exogenous $\varepsilon_t \mid I_{t-1} \sim N(0,\sigma^2)$

Regress $\hat{\varepsilon}_t^2$ on a constant and $\hat{\varepsilon}_{t-1}^2, \ldots \ldots, \hat{\varepsilon}_{t-q}^2$

$\Rightarrow \quad TR^2 \sim \chi^2 (q)$

Intuition : if ARCH effect is present $\quad \Rightarrow$ volatility clustering

$\Rightarrow$ past $\varepsilon_t^2$ predict current

Easy to use but <u>caution</u> because of regression in mean model misspecification

Alternative tests (also subject to caution) - use F - test (i.e. Wald)

- Portmanteau tests