QMLE

ENDOGEN. VAR: Y

EXOGEN. VAR: X

THE MODEL SPECIFIES THE CONDITIONAL DENSITY OF

$$Y_1 \dots Y_T \quad \text{GIVEN} \quad X_1 \dots X_T$$

IT IS PARAMETRIZED BY $\theta \in \Theta$, OPEN IN $\mathbb{R}^P$:

$$l(y_1 \dots y_T \mid x_1 \dots x_T ; \theta) = \prod_{i=1}^{T} f(y_i \mid x_i ; \theta), \qquad \theta \in \Theta$$

$Y_1 \dots Y_T$ are independent conditional on the exogen.

THE TRUE DENSITY OF Y:

$$l_0(y_1 \dots y_T \mid x_1 \dots x_T) = \prod_{i=1}^{T} f_0(y_i \mid x_i)$$

DOES NOT BELONG TO THE SELECTED FAMILY

$$f_0(y \mid x) \notin \{ f(y \mid x ; \theta), \ \theta \in \Theta \}$$

$\Rightarrow$ MISSPECIFICATION

MEASURE THE "DISTANCE" BTW THE TRUE f_0 AND THE MODEL $\{f(y \mid x ; \theta), \theta \in \Theta\}$ using KULLBACK CRITERION.

DEFINE THE PSEUDO-TRUE VALUE θ_0^* : THE VALUE OF CORRESPONDING TO THE MODEL CLOSEST POSSIBLE TO f_0.

$$\theta_0^* = \underset{\theta \in \Theta}{\operatorname{Arg\,max}} \; \underset{X}{E} \; \underset{0}{E} \; \log f(Y \mid X ; \theta)$$

E_0 : CONDITIONAL EXPECTATION OF Y GIVEN X UNDER f_0

SUPPOSE θ_0^* IS UNIQUE.

THE PSEUDO (QUASI) MAX LIKELIHOOD ESTIMATOR OF IS THE SOLUTION $\hat{\theta}_T$ OF

$$\underset{\theta \in \Theta}{\operatorname{MAX}} \sum_{i=1}^{T} \log f(Y_i \mid X_i ; \theta)$$

INTUITION

MLE COMPUTED FOR A MISSPECIFIED MODEL.

UNDER THE REGULARITY CONDITIONS QMLE CONVERGES TO TH PSEUDO-TRUE VALUE θ_0^*

THE MAXIMIZATION OF THE LIMITING PROBLEM IS:

$$\underset{\theta \in \Theta}{\operatorname{MAX}} \; \underset{X}{E} \; \underset{0}{E} \; \log f(Y \mid X ; \theta)$$

AND IT HAS A UNIQUE SOLUTION θ_0^*.

REGULARITY COND:

i) $Y_i \mid X_i$ are independent, with identical distribution.

ii) Θ is OPEN

iii) $\log f$ is CONTINUOUS IN θ AND INTEGRABLE W.R TO THE TRUE DENSITY f_0, $\forall \theta$.

iv) $\frac{1}{T} \sum_{i=1}^{T} \log f(Y_i \mid X_i ; \theta)$ CONVERGE A.S. UNIFORMLY

ON Θ TO $E_X E_0 \log f(Y \mid X ; \theta)$

E_X : expectation w.r to the true marginal density of X

E_0 : expectation w.r to the true conditional density of $Y \mid X$

v) THE LIMITING PROBLEM ADMITS A UNIQUE SOLUTION = PSEUDO TRUE VALUE

ASYMPTOTIC NORMALITY

* DEVELOP IN THE NEIGHBORHOOD OF THE LIMITING PSEUDO TRUE VALUE:

$$\sum_{i=1}^{T} \frac{\partial \log f(Y_i \mid X_i ; \hat{\theta}_T)}{\partial \theta} = 0$$

$$\Rightarrow \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \frac{\partial \log f(Y_i \mid X_i ; \tilde{\theta}_0^*)}{\partial \theta} + \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \frac{\partial^2 \log f}{\partial \theta \, \partial \theta'} \left[\hat{\theta}_T - \tilde{\theta}_0^* \right] \simeq 0$$

(4)

$$\sqrt{T}\left[\hat{\theta}_T - \tilde{\theta}_0^*\right] \# \left[-\frac{1}{T}\sum_{i=1}^{T}\frac{\partial^2 \log f}{\partial\theta\,\partial\theta'}\right]^{-1}\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\frac{\partial \log f}{\partial\theta}$$

$$\# \left[\underset{X}{E}\,\underset{0}{E} - \frac{\partial^2 \log f}{\partial\theta\,\partial\theta'}\right]^{-1}\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\frac{\partial \log f}{\partial\theta}$$

Law of Large No for the empirical mean of 2nd derivatives

$\#$: difference $\rightarrow 0$ in probability

$$\underset{X}{E}\,\underset{0}{E}\;\frac{\partial \log f\left[Y_i \mid X_i ; \tilde{\theta}_0^*\right]}{\partial\theta} = \frac{\partial}{\partial\theta}\,\underset{X}{E}\,\underset{0}{E}\,\log f(Y_i \mid X_i, \theta_0^*) = 0$$

since θ_0^* is the solution of the limiting problem

THE VECTORS $\dfrac{\partial \log f(Y_i \mid X_i ; \tilde{\theta}_0^*)}{\partial\theta}$ are iid of mean 0

and var-covar:

$$I = \underset{X}{E}\,\underset{0}{E}\left[\frac{\partial \log f(Y \mid X ; \theta_0^*)}{\partial\theta}\;\frac{\partial \log f(Y \mid X ; \theta_0^*)}{\partial\theta}\right]$$

THE CENTRAL LIMIT THEOREM $\Rightarrow$

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{T}\frac{\partial \log f(Y_i \mid X_i ; \tilde{\theta}_0^*)}{\partial\theta} \underset{ASY}{\sim} N[0, I]$$

THE VECTOR $\sqrt{T}\left[\tilde{\theta}_T - \theta_0^*\right]$ is A LINEAR TRANSFORM OF IT;

FOLLOWS A $N(0, J^{-1} I J^{-1})$

$$J = \underset{X}{E}\ \underset{0}{E}\left[-\frac{\partial^2 \log f(Y \mid X; \theta_0^*)}{\partial\theta\, \partial\theta'}\right]$$

UNDER MISSPECIFICATION $I \neq J$

$$\hat{J} = -\frac{1}{T}\sum_{i=1}^{T} \frac{\partial^2 \log f(Y_i \mid X_i ; \hat{\theta}_T)}{\partial\theta\, \partial\theta'}$$

$$\hat{I} = \frac{1}{T}\sum_{i=1}^{T}\left[\frac{\partial \log f(Y_i \mid X_i ; \hat{\theta}_T)}{\partial\theta}\ \frac{\partial \log f(Y_i \mid X_i ; \hat{\theta}_T)}{\partial\theta'}\right]$$