STOCHASTIC TREND MODELS

- RANDOM WALK

$$y_t = y_{t-1} + \varepsilon_t$$

ε_t is A STRONG WHITE NOISE $\Rightarrow$ I.I.D SEQUENCE

y_0 is a constant

$$y_t = y_0 + \underbrace{\sum_{i=1}^{t} \varepsilon_i}_{\text{stochastic trend}}$$

$\text{var } y_t = t\sigma^2 \to \infty$ as $t \to \infty$

linear decay of ACF

$$\Delta y_t = \varepsilon_t$$

RANDOM WALK + DRIFT

$$y_t = y_{t-1} + a_0 + \varepsilon_t$$

$$y_t = y_0 + \underbrace{a_0 t}_{\text{linear deterministic trend}} + \underbrace{\sum_{i=1}^{t} \varepsilon_i}_{\text{stochastic trend.}}$$

$$E_t\, y_{t+s} = y_t + a_0 s$$

$$\Delta y_t = a_0 + \varepsilon_t$$

• RANDOM WALK + NOISE

$$y_t = \mu_t + \eta_t$$

$$\mu_t = \mu_{t-1} + \varepsilon_t$$

η_t WN with variance σ_η^2

$\varepsilon_t, \eta_{t-s}$ independent $\forall s, t$

μ_0 : initial condition = const.

$$\mu_t = \mu_0 + \sum_{i=1}^{t} \varepsilon_i$$

$$y_t = \mu_0 + \sum_{i=1}^{t} \varepsilon_i + \eta_t$$

$$= y_0 - \eta_0 + \underbrace{\sum \varepsilon_i}_{\text{stoch trend}} + \eta_t$$

$y_0 = \mu_0 + \eta_0$

$\text{var } y_t = t\sigma^2 + \sigma_\eta^2$

$$\Delta y_t = \varepsilon_t + \Delta\eta_t$$

$$= \varepsilon_t + (\eta_t - \eta_{t-1}) \quad \text{MA(1)}$$

- GENERAL TREND + IRREGULAR

$$y_t = \mu_t + \eta_t$$

$$\mu_t = \mu_{t-1} + a_0 + \varepsilon_t$$

① $$y_t = \underbrace{y_0 - \eta_0 + a_0 t}_{\text{deterministic trend}} + \underbrace{\sum_{i=1}^{t} \varepsilon_i}_{\text{stochastic trend}} + \eta_t$$

or: or: $+ \underbrace{\eta_t - \beta_1 \eta_{t-1} \cdots}_{\text{irregular}}$

Δy_t is MA(1)

- LOCAL LINEAR TREND

$$y_t = \mu_t + \eta_t$$

$$\mu_t = \mu_{t-1} + a_t + \varepsilon_t$$

$$a_t = a_{t-1} + \delta_t \qquad \{\delta_t\} : WN$$

$\Delta^2 y_t$ is MA(2)

PROCESSES

DS	TS
DIFFERENCE STATIONARY	TREND STATIONARY
• AFTER DIFFERENCING IS STATIONARY	• AFTER DIFFERENCING IS NON-INVERTIBLE
• REMOVING DETER. TREND ⇒ STOCHASTIC TREND REMAING	• REMOVING DETER TREND ⇒ STATIONARY

• OVERDIFFERENCING : noninvertible

Table 3.2 **Selected Autocorrelations from Nelson and Plosser**

	$\rho(1)$	$\rho(2)$	r(1)	r(2)	d(1)	d(2)
Real GNP	0.95	0.90	0.34	0.04	0.87	0.66
Nominal GNP	0.95	0.89	0.44	0.08	0.93	0.79
Industrial production	0.97	0.94	0.03	–0.11	0.84	0.67
Unemployment rate	0.75	0.47	0.09	–0.29	0.75	0.46

Notes: 1. Full details of the correlogram can be obtained from Nelson and Plosser (1982) who report the first six sample autocorrelations.
2. Respectively, $\rho(i)$, $r(i)$, and $d(i)$ refer to the *i*th-order autocorrelation coefficient of each series, first difference of the series, and detrended values of the series.

are generated from DS processes. Nelson and Plosser point out that the positive autocorrelation of differenced real and nominal GNP at lag 1 only is suggestive of an MA(1) process. To further strengthen the argument for DS-generating processes, recall that differencing a TS process yields a noninvertible moving process. None of the differenced series reported by Nelson and Plosser appear to have a unit root in the MA terms.

The results from fitting a linear trend to the data and forming sample autocorrelations of the residuals are shown in the last two columns of the table. An interesting feature of the data is that the sample autocorrelations of the detrended data are reasonably high. This is consistent with the fact that detrending a DS series will not eliminate the nonstationarity. Notice that detrending the unemployment rate has *no effect* on the autocorrelations.

Rather than rely solely on an analysis of correlograms, it is possible to formally test whether a series is difference stationary. We examine such formal tests in the next chapter. The testing procedure is not as straightforward as it might seem. We cannot use the usual statistical techniques since classical procedures all presume that the data are stationary. For now, it suffices to say that Nelson and Plosser are not able to reject the null hypothesis that their data are DS. If this view is correct, macroeconomic variables do not grow at a smooth long-run rate. Some macroeconomic shocks are of a permanent nature; the effects of such shocks are never eliminated.

11. STOCHASTIC TRENDS AND UNIVARIATE DECOMPOSITIONS

Nelson and Plosser's (1982) findings suggest that many economic time series have a stochastic trend and an irregular component. Having observed a series, but not the individual components, is there any way to decompose the series into the con-

$t=2$

to test the hypotheses $\gamma = 0$. Dickey
tatistics (called ϕ_1, ϕ_2 and ϕ_3) to test
10) or (4.13), the null hypothesis $\gamma =$
g a time trend in the regression—so
pothesis $a_0 = \gamma = a_2 = 0$ is tested us-
$a_2 = 0$ is tested using the ϕ_3 statistic.
in exactly the same way as ordinary

S(unrestricted)] r
ed)/$(T-k)$

= the sums of the squared residuals
odels

Model	Hypothesis	Test Statistic	Critical values for 95% and 99% Confidence Intervals
$\Delta y_t = a_0 + \gamma y_{t-1} + a_2 t + \epsilon_t$	$\gamma = 0$	τ_τ	−3.45 and −4.04
	$a_0 = 0$ given $\gamma = 0$	$\tau_{\alpha\tau}$	3.11 and 3.78
	$a_2 = 0$ given $\gamma = 0$	$\tau_{\beta\tau}$	2.79 and 3.53
	$\gamma = a_2 = 0$	ϕ_3	6.49 and 8.73
	$a_0 = \gamma = a_2 = 0$	ϕ_2	4.88 and 6.50
$\Delta y_t = a_0 + \gamma y_{t-1} + \epsilon_t$	$\gamma = 0$	τ_μ	−2.89 and −3.51
	$a_0 = 0$ given $\gamma = 0$	$\tau_{\alpha\mu}$	2.54 and 3.22
	$a_0 = \gamma = 0$	ϕ_1	4.71 and 6.70
$\Delta y_t = \gamma y_{t-1} + \epsilon_t$	$\gamma = 0$	τ	−1.95 and −2.60

Notes: Critical values are for a sample size of 100.

$$\phi_i = \frac{\text{RSS (restricted)} - \text{RSS (unrestricted)}/r}{\text{RSS (unrest)}/(T-K)}$$

$$= \frac{(\hat{\sigma}_R^2 - \hat{\sigma}_U)/\text{\# of param}}{\hat{\sigma}_U/(T-K)}$$

UNIT ROOT TESTS

D.F: $\underline{AR(1)}$

① $x_t = \rho x_{t-1} + e_t \Rightarrow \hat{\tau} = T(\hat{\rho} - 1)$

- $\Delta x_t = \underline{(\rho - 1)} x_{t-1} + e_t \Rightarrow \hat{\tau}$.

② $x_t = \alpha + \rho x_{t-1} + e_t \Rightarrow \hat{\rho}_\mu$

- $\Delta x_t = \alpha + \underline{(\rho - 1)} x_{t-1} + \varepsilon_t \Rightarrow \hat{\tau}_\mu$

③ $x_t = \alpha + \beta t + \rho x_{t-1} + \varepsilon_t \Rightarrow \hat{\rho}_\tau$

- $\Delta x_t = \alpha + \beta t + \underline{(\rho - 1)} x_{t-1} + \varepsilon_t \Rightarrow \hat{\tau}_\tau$

ADF: $\underline{AR(p)}$

① $\Delta x_t = (\rho_1 - 1) x_{t-1} + \sum_{i=2}^{p} \rho_i \Delta x_{t-i+1} + e_t$

Similarly plotted

② $\Delta x_t = \alpha + (\rho_1 - 1) x_{t-1} + \sum_{i=2}^{p} \rho_i \Delta x_{t-i+1} + e_t$

$\hat{\tau}_\mu$

③ $\Delta x_t = \alpha + \beta t + (\rho_1 - 1) x_{t-1} + \sum_{i=2}^{p} \rho_i \Delta x_{t-i+1} + e_t$

$\hat{\tau}_\tau$

ARMA

$\Delta x_t = (\rho - 1) x_{t-1} + \sum_{i=2}^{K} \beta_i \Delta x_{t-i+1} + e_t$, where $\underline{K = T^{1/3}}$

only $\hat{\tau}$

Table 2.1

Tests for a Unit Root in Selected Financial Series

Series	$\hat{\rho}$	DF	ADF(4)	5% crit value
log share pr	.999	.11	.007	
log int rate	.975	−3.016	−3.113	−2.867
log $US/SF	.996	−1.318	−1.5636	−2.866
log returns	.111	−24.7492	−11.7512	−2.8657

Series are described in section 1. ADF(4) is augmented Dickey Fuller Test with 4 lags.

	$\gamma_1 = 0$	$\gamma_2 = 0$	$\gamma_5 = \gamma_6 = 0$
Intercept	–2.88	–1.95	3.08
Intercept plus seasonal dummies	–2.95	–2.94	6.57
Intercept plus seasonal dummies plus time trend	–3.53	–2.94	6.60

4. EXAMPLES OF THE AUGMENTED DICKEY–FULLER TEST

The last chapter reviewed the evidence reported by Nelson and Plosser (1982) suggesting that macroeconomic variables are difference stationary rather than trend stationary. We are now in a position to consider their formal tests of the hypothesis. For each series under study, Nelson and Plosser estimated the regression:

$$\Delta y_t = a_0 + a_2 t + \gamma y_{t-1} + \sum_{i=2}^{p} \beta_i \Delta y_{t-1+i} + \epsilon_t$$

The chosen lag lengths are reported in the column labeled p in Table 4.2. The estimated values a_0, a_2, and γ are reported in columns 3, 4, and 5, respectively.

Table 4.2 **Nelson and Plosser's Tests for Unit Roots**

	p	a_0	a_2	γ	$\gamma + 1$
Real GNP	2	0.819 (3.03)	0.006 (3.03)	–0.175 (–2.99)	0.825
Nominal GNP	2	1.06 (2.37)	0.006 (2.34)	–0.101 (–2.32)	0.899
Industrial production	6	0.103 (4.32)	0.007 (2.44)	–0.165 (–2.53)	0.835
Unemployment rate	4	0.513 (2.81)	–0.000 (–0.23)	–0.294* (–3.55)	0.706

Notes:
1. p is the chosen lag length. Coefficients divided by their standard errors are in parentheses. Thus, entries in parentheses represent the t-test for the null hypothesis that a coefficient is equal to zero. Under the null of nonstationary, it is necessary to use the Dickey–Fuller critical values. At the 0.05 significance level, the critical value for the t-statistic is –3.45.
2. An asterisk (*) denotes significance at the 0.05 level. For real and nominal GNP and industrial production, it is not possible to reject the null $\gamma = 0$ at the 0.05 level. Hence, the unemployment rate appears to be stationary.
3. The expression $\gamma + 1$ is the estimate of the partial autocorrelation between y_t and y_{t-1}.

TABLE B.5
Critical Values for the Phillips-Perron Z_ρ Test and for the Dickey-Fuller Test Based on Estimated *OLS* Autoregressive Coefficient

Sample size T	*Probability that $T(\hat{\rho}-1)$ is less than entry*							
	0.01	*0.025*	*0.05*	*0.10*	*0.90*	*0.95*	*0.975*	*0.99*
				Case 1				
25	−11.9	−9.3	−7.3	−5.3	1.01	1.40	1.79	2.28
50	−12.9	−9.9	−7.7	−5.5	0.97	1.35	1.70	2.16
100	−13.3	−10.2	−7.9	−5.6	0.95	1.31	1.65	2.09
250	−13.6	−10.3	−8.0	−5.7	0.93	1.28	1.62	2.04
500	−13.7	−10.4	−8.0	−5.7	0.93	1.28	1.61	2.04
∞	−13.8	−10.5	−8.1	−5.7	0.93	1.28	1.60	2.03
				Case 2				
25	−17.2	−14.6	−12.5	−10.2	−0.76	0.01	0.65	1.40
50	−18.9	−15.7	−13.3	−10.7	−0.81	−0.07	0.53	1.22
100	−19.8	−16.3	−13.7	−11.0	−0.83	−0.10	0.47	1.14
250	−20.3	−16.6	−14.0	−11.2	−0.84	−0.12	0.43	1.09
500	−20.5	−16.8	−14.0	−11.2	−0.84	−0.13	0.42	1.06
∞	−20.7	−16.9	−14.1	−11.3	−0.85	−0.13	0.41	1.04
				Case 4				
25	−22.5	−19.9	−17.9	−15.6	−3.66	−2.51	−1.53	−0.43
50	−25.7	−22.4	−19.8	−16.8	−3.71	−2.60	−1.66	−0.65
100	−27.4	−23.6	−20.7	−17.5	−3.74	−2.62	−1.73	−0.75
250	−28.4	−24.4	−21.3	−18.0	−3.75	−2.64	−1.78	−0.82
500	−28.9	−24.8	−21.5	−18.1	−3.76	−2.65	−1.78	−0.84
∞	−29.5	−25.1	−21.8	−18.3	−3.77	−2.66	−1.79	−0.87

The probability shown at the head of the column is the area in the left-hand tail.

Source: Wayne A. Fuller, *Introduction to Statistical Time Series*, Wiley, New York, 1976, p. 371.

Table 8.5.2. Empirical cumulative distribution of $\hat{\tau}$ for $\rho = 1$

Sample Size	Probability of a Smaller Value							
n	0.01	0.025	0.05	0.10	0.90	0.95	0.975	0.99
				$\hat{\tau}$				
25	−2.66	−2.26	−1.95	−1.60	0.92	1.33	1.70	2.16
50	−2.62	−2.25	−1.95	−1.61	0.91	1.31	1.66	2.08
100	−2.60	−2.24	−1.95	−1.61	0.90	1.29	1.64	2.03
250	−2.58	−2.23	−1.95	−1.62	0.89	1.29	1.63	2.01
500	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
∞	−2.58	−2.23	−1.95	−1.62	0.89	1.28	1.62	2.00
				$\hat{\tau}_\mu$				
25	−3.75	−3.33	−3.00	−2.63	−0.37	0.00	0.34	0.72
50	−3.58	−3.22	−2.93	−2.60	−0.40	−0.03	0.29	0.66
100	−3.51	−3.17	−2.89	−2.58	−0.42	−0.05	0.26	0.63
250	−3.46	−3.14	−2.88	−2.57	−0.42	−0.06	0.24	0.62
500	−3.44	−3.13	−2.87	−2.57	−0.43	−0.07	0.24	0.61
∞	−3.43	−3.12	−2.86	−2.57	−0.44	−0.07	0.23	0.60
				$\hat{\tau}_\tau$				
25	−4.38	−3.95	−3.60	−3.24	−1.14	−0.80	−0.50	−0.15
50	−4.15	−3.80	−3.50	−3.18	−1.19	−0.87	−0.58	−0.24
100	−4.04	−3.73	−3.45	−3.15	−1.22	−0.90	−0.62	−0.28
250	−3.99	−3.69	−3.43	−3.13	−1.23	−0.92	−0.64	−0.31
500	−3.98	−3.68	−3.42	−3.13	−1.24	−0.93	−0.65	−0.32
∞	−3.96	−3.66	−3.41	−3.12	−1.25	−0.94	−0.66	−0.33

This table was constructed by David A. Dickey using the Monte Carlo method. Details are given in Dickey (1975). Standard errors of the estimates vary, but most are less than 0.02.

To extend the results for the first order process with $\rho = 1$ to the pth order autoregressive process, we consider the time series

$$Y_t = \sum_{j=1}^{t} Z_j, \qquad t = 1, 2, \ldots, \tag{8.5.11}$$

where $\{Z_t \colon t \in (0, \pm 1, \pm 2, \ldots)\}$ is a $(p-1)$ order autoregressive time series with the representation

$$Z_t + \sum_{i=2}^{p} a_i Z_{t-i+1} = e_t, \tag{8.5.12}$$