# MULTIVARIATE TIME SERIES

$$Y_t = [Y_{1t} \quad Y_{2t} \quad \dots \quad Y_{nt}]'$$

ex: $Y_t = \begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix}$

has marginal mean:

$$m = E(Y_t) = \begin{bmatrix} E(Y_{1t}) \\ E(Y_{2t}) \end{bmatrix}$$

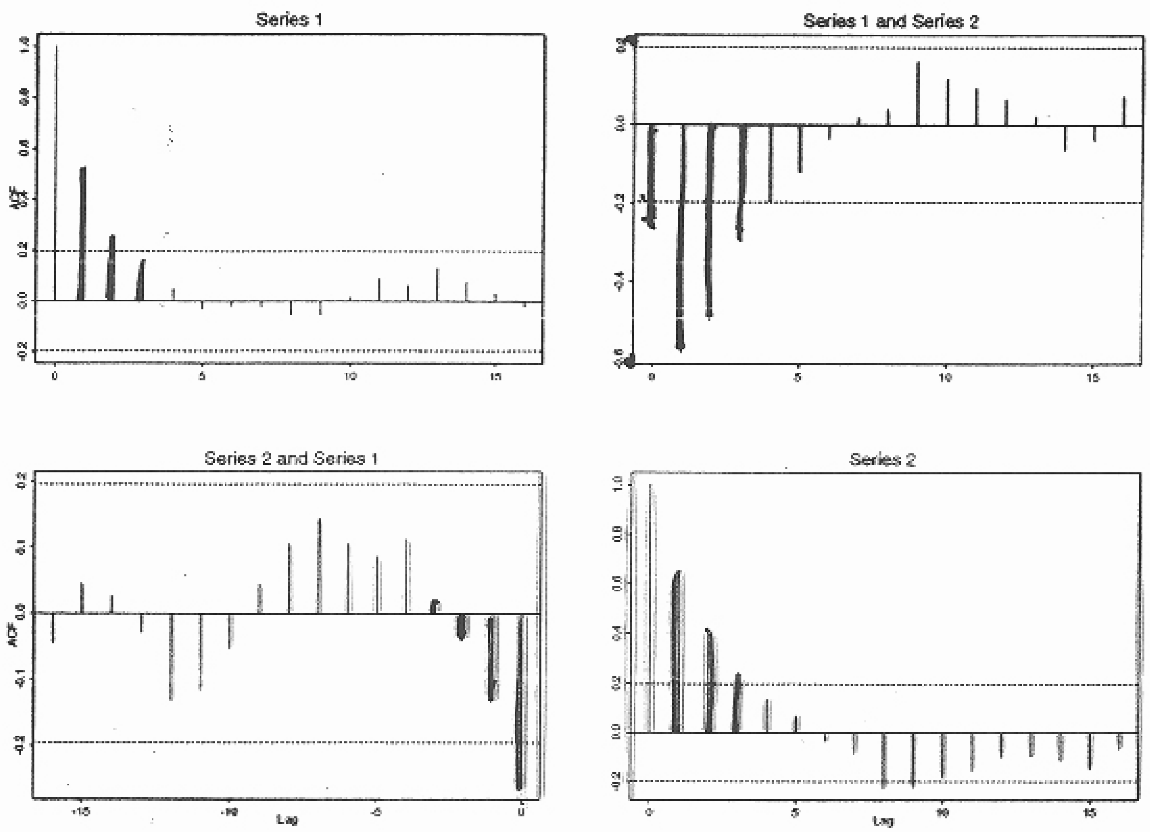
and marginal var-covar

$$V(Y_t) = \begin{bmatrix} V(Y_{1t}) & cov(Y_{1t}\ Y_{2t}) \\ cov(Y_{2t}\ Y_{1t}) & V(Y_{2t}) \end{bmatrix} = \Gamma(0)$$

SYMMETRIC, POSITIVE DEF. i.e.

$$a' V(Y_t) a = V(a' Y_t) > 0 \qquad \forall a \text{ real}, [n]$$

Figure 3.2: Autocorrelations, positive real eigenvalues



MULTIVARIATE AUTOCOVARIANCE:

$$\Gamma(h) = cov(Y_t, Y_{t-h}) = E(Y_t Y_{t-h}') - E(Y_t) E(Y_{t-h}')$$

$$\Gamma(h) = \begin{bmatrix} cov(Y_{1t} \; Y_{1t-h}) & cov(Y_{1t} \; Y_{2t-h}) \\ cov(Y_{2t}, Y_{1t-h}) & cov(Y_{2t}, Y_{2t-h}) \end{bmatrix}$$

NOT SYMMETRIC FOR $h \neq 0$. HOWEVER: $\Gamma(-h) = \Gamma(h)'$.

$$Corr(Y_{1t-h}, Y_{2t}) = \frac{cov(Y_{1t-h}, Y_{2t})}{\sqrt{V(Y_{1t-h})} \sqrt{V(Y_{2t})}}$$

$$corr(0) = \begin{bmatrix} 1 & corr(Y_{1t} \; Y_{2t}) \\ corr(Y_{1t} \; Y_{2t}) & 1 \end{bmatrix}, \quad corr(h) = \begin{bmatrix} corr(Y_{1t} \; Y_{1t-h}) & corr(Y_{1t} \; Y_{2t-h}) \\ corr(Y_{2t} \; Y_{1t-h}) & corr(Y_{2t}, Y_{2t-h}) \end{bmatrix}$$

ACF $\equiv$ SEQ OF MATRICES AT $h = 0, \pm 1, \pm 2 \ldots$

- SECOND ORDER STATIONARITY FOR MULTIVARIATE

1) mean $m$ time independent

2) autocovariance $\Gamma(h)$ independent of $t$, depends on $h$ only

- WEAK WHITE MULTIVARIATE NOISE

1) $m = 0$

2) $\Gamma(h) = 0 \quad \forall h \neq 0$, SERIALLY UNCORRELATED

ex: $\varepsilon_t$ with var-covar: $\Omega = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

Standard w. noise $\tilde{\varepsilon}_t$ with var-covar $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\tilde{\varepsilon}_t = \Omega^{-\frac{1}{2}} \varepsilon_t$$

- ESTIMATORS:

$$\hat{m}_T = \frac{1}{T} \sum_{t=1}^{T} Y_t = \begin{bmatrix} \frac{1}{T} \sum Y_{1t} \\ \frac{1}{T} \sum Y_{2t} \end{bmatrix}$$

$$\hat{\Gamma}(h) = \frac{1}{T} \sum_{t=h+1}^{T} (Y_t - \hat{m}_T)(Y_{t-h} - \hat{m}_T)' = \begin{bmatrix} \hat{\gamma}_{11}(h) & \hat{\gamma}_{12}(h) \\ \hat{\gamma}_{21}(h) & \hat{\gamma}_{22}(h) \end{bmatrix}$$

ACF:

$$\text{correl}(h) = \begin{bmatrix} \hat{\rho}_{11}(h) & \hat{\rho}_{12}(h) \\ \hat{\rho}_{21}(h) & \hat{\rho}_{22}(h) \end{bmatrix}$$

$$\hat{\rho}_{11}(h) = \frac{\hat{\gamma}_{11}(h)}{\hat{\gamma}_{11}(0)} \quad , \quad \hat{\rho}_{12}(h) = \frac{\hat{\gamma}_{12}(h)}{\sqrt{\hat{\gamma}_{11}(0)}\sqrt{\hat{\gamma}_{22}(0)}}$$

MULTIVARIATE MODELS:

- VECTOR AUTOREGRESSIVE : VAR(1)

$$Y_t = \Phi Y_{t-1} + \varepsilon_t \implies \begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} Y_{1t-1} \\ Y_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

$$\begin{cases} Y_{1t} = \phi_{11} Y_{1t-1} + \phi_{12} Y_{2t-1} + \varepsilon_{1t} \\ Y_{2t} = \phi_{21} Y_{1t-1} + \phi_{22} Y_{2t-1} + \varepsilon_{2t} \end{cases}$$

$\Phi$ is an $(n \times n)$ MATRIX WITH EIGENVALUES LESS THAN ONE IN ABSOLUTE VALUES

$\{\varepsilon_t\}$ : n dimensional weak WHITE NOISE $\sim (0, \Omega)$

CAN BE WRITTEN

$$Y_t = \Phi Y_{t-1} + \varepsilon_t = \Phi^2 Y_{t-2} + \varepsilon_t + \Phi \varepsilon_{t-1}$$

$$\vdots$$

$$= \Phi^h Y_{t-h} + \varepsilon_t + \Phi \varepsilon_{t-1} + \dots + \Phi^{h-1} \varepsilon_{t-h+1}$$

(VMA$(\infty)$

$$= \varepsilon_t + \Phi \varepsilon_{t-1} + \dots + \Phi^h \varepsilon_{t-h} + \dots = \sum_{h=0}^{\infty} \Phi^h \varepsilon_{t-h}$$

marginal moments:

- $E(Y_t) = 0$

- $V(Y_t) = \Gamma(0) = V\left[\sum_{h=0}^{\infty} \Phi^h \varepsilon_{t-h}\right]$

$$= \sum \Phi^h V(\varepsilon_{t-h}) \Phi'^h$$

$$= \sum_{h=0}^{\infty} \Phi^h \Omega \Phi'^h$$

$$\mathrm{cov}(Y_t, Y_{t-h}) = \mathrm{cov}(\Phi^h Y_{t-h} + \varepsilon_t + \Phi \varepsilon_{t-1} + \Phi^{h-1} \varepsilon_{t-h+1}, Y_{t-h})$$

$$= \mathrm{cov}(\Phi^h Y_{t-h}, Y_{t-h})$$

- $\Gamma(h) = \Phi^h \Gamma(0)$

CONDITIONAL MEAN

$$E(Y_t \mid Y_{t-1}, \ldots) = E(Y_t \mid Y_{t-1}) = \Phi\, Y_{t-1}$$

COND VAR:

$$E\left[ [Y_t - E(Y_t \mid Y_{t-1})][Y_t - E(Y_t \mid Y_{t-1})]' \mid Y_{t-1} \right] = E(\varepsilon_t \varepsilon_t') = \Omega$$

- FORECAST FOR A SAMPLE OVER $(1, 2, \ldots T)$:

at step 1: $E[Y_{T+1} \mid Y_T] = \Phi\, Y_T$

h step ahead: $E[Y_{T+h} \mid Y_T] = \Phi^h\, Y_T$

- WHAT ARE THE COMPONENT PROCESSES IN VAR(1) ?

→ in general ARMA$(n, n-1)$

ex:

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} Y_{1t-1} \\ Y_{2t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

$$Y_t = \Phi Y_{t-1} + \varepsilon_t \Rightarrow (I - \Phi L) Y_t = \varepsilon_t$$

$$\begin{bmatrix} 1-\phi_{11}L & -\phi_{12}L \\ -\phi_{21}L & 1-\phi_{22}L \end{bmatrix} \begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \varepsilon_t$$

$$\begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{bmatrix} 1-\phi_{11}L & -\phi_{12}L \\ -\phi_{21}L & 1-\phi_{22}L \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

$$= \frac{1}{(1-\phi_{11}L)(1-\phi_{22}L) - \phi_{12}L^2} \begin{bmatrix} 1-\phi_{22}L & \phi_{12}L \\ \phi_{21}L & 1-\phi_{11}L \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

ALLOWS AN EXPANSION IN PAST $\varepsilon_t$'s WHICH IS NOT EXPLOSIVE $\Leftrightarrow$ THE ROOTS $L_1$ and $L_2$ of this $2^{nd}$ order polynomial ARE OUTSIDE THE UNIT CIRCLE, i.e. GREATER THAN 1 in A.V.

MULTIVARIATE WOLD THEOREM:

ANY 2nd ORDER STATIONARY $Y_t$ ADMITS A VMA($\infty$):

$$Y_t = \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \Theta_2 \varepsilon_{t-2} + \dots$$

WHERE $\varepsilon_t$ n-dim WN.

OTHER LINEAR PROCESSES

VAR(p) : $Y_t = \Phi_1 Y_{t-1} + \cdots + \Phi_p Y_{t-p} + \varepsilon_t$

VMA(q): $Y_t = \varepsilon_t + \Theta_1 \varepsilon_{t-1} + \cdots + \Theta_q \varepsilon_{t-q}$

VARMA(p,q) : $Y_t = \Phi_1 Y_{t-1} + \cdots + \Phi_p Y_{t-p} + \Theta_1 \varepsilon_{t-1} + \cdots + \Theta_q \varepsilon_{t-q}$

$$\Phi(L)\, Y_t = \Theta(L)\, \varepsilon_t$$

$\Phi(L) = I - \Phi_1 L - \cdots - \Phi_p L^p$

$\Theta(L) = 1 + \Theta_1 L + \cdots + \Theta_q L^p$

THE VMA($\infty$) $Y_t = \Phi(L)^{-1}\, \Theta(L)\, \varepsilon_t$

and VAR($\infty$) $\Theta(L)^{-1}\, \Phi(L)\, Y_t = \varepsilon_t$

YIELD CONVERGING SERIES IN NONNEG POWERS OF L $\Leftrightarrow$

- STATIONARITY : roots of $\det \Phi(x) = 0$ greater than 1 in abs value.

- INVERTIBILITY : roots of $\det \Theta(x) = 0$ greater than 1 in abs. value

# ESTIMATION (SEEMINGLY UNRELATED EQ)

- RECALL THE SUR MODEL

$$Y_{1t} = \beta_{11} X_{1t} + \dots + \beta_{1K} X_{Kt} + \varepsilon_{1t}$$
$$Y_{2t} = \beta_{21} X_{1t} + \dots + \beta_{2K} X_{Kt} + \varepsilon_{2t}$$

$$Y_t = B X_t + \varepsilon_t$$

IF
THE SAME REGRESSORS ON THE R.H.S OF EACH EQUATIONS

THAN GLS $\equiv$ OLS FROM EACH EQUATION.

DESPITE THAT THE ERRORS ARE CONTEMPORANEOUSLY CORRELATED

I.E. IN A SAMPLE OF T obs:

$$W = \begin{bmatrix} \sigma_1^2 I_T & \sigma_{12} I_T \\ \sigma_{12} I_T & \sigma_2^2 I_T \end{bmatrix}$$

RECALL: $GLS = (X' \hat{W}^{-1} X)^{-1} X' \hat{W}^{-1} Y$

$$VAR(1): \begin{bmatrix} Y_{1t} \\ Y_{2t} \end{bmatrix} = \begin{matrix} \phi_{11} Y_{1t-1} + \phi_{12} Y_{2t-1} + \varepsilon_{1t} \\ \phi_{21} Y_{1t-1} + \phi_{22} Y_{2t-1} \quad \varepsilon_{2t} \end{matrix}$$