Supplement to "Composite Likelihood for Stochastic Migration Model with Unobserved Factor"*

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August 2020

Abstract

This document contains the appendices for the authors' paper "Composite Likelihood for Stochastic Migration Model with Unobserved Factor". It provides proofs, simulation details, and additional simulation results.

Keywords: Migration Model, Credit Rating, Basel III, Composite Likelihood, Factor Model, Large Panel.

*The authors gratefully acknowledge financial support of the chair ACPR/Risk Foundation: Regulation and Systemic Risks, the ECR DYSMOIA and the Natural Sciences and Engineering Council of Canada.

Appendix A: The Expected Transition Matrices

A.1. Expected Matrix P (Lemma 1)

We have:

$$y^*_{i,t} = \beta_l f_t + \delta_l + \sigma_l u_{i,t}, \text{ if } \ y_{i,t-1} = l,$$

where $u_{i,t} \sim N(0,1)$ and $f_t \sim N(0,1)$ are independent. Then, if $y_{i,t-1} = l$, $y^*_{i,t} \sim N(\delta_l, \sigma_l^2 + \beta_l^2)$. It follows that:

$$P[y_{i,t} = k | y_{i,t-1} = l] = P[c_k < y^*_{i,t} < c_{k+1} | y_{i,t-1} = l]$$

and

$$p_{kl}(\theta) = \Phi\left(\frac{c_{k+1} - \delta_l}{\sqrt{\sigma_l^2 + \beta_l^2}}\right) - \Phi\left(\frac{c_k - \delta_l}{\sqrt{\sigma_l^2 + \beta_l^2}}\right).$$

A.2. Matrix $P(2)$ (Lemma 2)

We have:

$$P(2) = E[P(f_t;\theta)\ P(f_{t-1};\theta)] = E[P(\rho f_{t-1} + \sqrt{1-\rho^2}\ \eta_t;\theta)]\ P(f_{t-1};\theta)].$$

Since f_{t-1} and η_t are independent, $\eta_t \sim N(0,1)$ and $f_{t-1} \sim N(0,1)$, we get:

$$\begin{aligned} P(2) &= E_{f_{t-1}}\ E_{\eta_t}\left[P(\rho f_{t-1} + \sqrt{1-\rho^2}\ \eta_t;\theta)\ P(f_{t-1};\theta) | f_{t-1}\right], \\ &= E_{f_{t-1}}\left[E_{\eta_t}\left[P(\rho f_{t-1} + \sqrt{1-\rho^2}\ \eta_t;\theta) | f_{t-1}\right]\ P(f_{t-1};\theta)\right], \\ &= E_{f_{t-1}}\left[A\ B\right], \end{aligned}$$

where the components of matrix A are given by:

$$\begin{aligned} a_{kl}(f_{t-1};\theta,\rho) &= \mathbb{P}\left[c_k < y^*_{i,t} < c_{k+1} |\ y_{i,t-1} = l, f_{t-1}\right] \\ &= \mathbb{P}\left[c_k < \delta_l + \beta_l \rho f_{t-1} + \beta_l \sqrt{1-\rho^2}\ \eta_t + \sigma_l u_{i,t} < c_{k+1} | y_{1,t-1}, f_{t-1}\right], \\ &= \Phi\left(\frac{c_{k+1} - \delta_l - \beta_l \rho f_{t-1}}{\sqrt{\sigma_l^2 + \beta_l^2(1-\rho^2)}}\right) - \Phi\left(\frac{c_k - \delta_l - \beta_l \rho f_{t-1}}{\sqrt{\sigma_l^2 + \beta_l^2(1-\rho^2)}}\right), k, l, = 1, ...K, \end{aligned}$$

by the independence between $(\eta_t, u_{i,t})$ and $(y_{i,t-1}, f_{t-1})$. By (2.4) the elements of matrix B are:

$$p_{kl}(f_{t-1};\theta) = \Phi\left(\frac{c_{k+1}-\delta_l-\beta_l f_{t-1}}{\sigma_l}\right) - \Phi\left(\frac{c_k-\delta_l-\beta_l f_{t-1}}{\sigma_l}\right), k,l=1,...,K.$$

Therefore, by integrating out f_{t-1}, we get:

$$\begin{aligned} p_{kl}(2;\theta,\rho) &= \int \sum_{j=1}^{K} [a_{k,j}(f;\theta,\rho)\ p_{j,l}(f;\theta)]\phi(f)df \\ &= \int \sum_{j=1}^{K} \left[\Phi\left(\frac{c_{k+1}-\delta_j-\beta_j\rho f}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right) - \Phi\left(\frac{c_k-\delta_j-\beta_j\rho f}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right)\right] \\ &\times \left[\Phi\left(\frac{c_{j+1}-\delta_l-\beta_l f}{\sigma_l}\right) - \Phi\left(\frac{c_j-\delta_l-\beta_l f}{\sigma_l}\right)\right]\phi(f)df. \end{aligned}$$

Appendix B: Proof of Propositions 1 and 2

B.1. Proof of Proposition 1

From Lemma 1, and the definitions $c_1 = -\infty$, $c_{K+1} = \infty$, we know that the identifying functions of parameters are:

$$\frac{c_k-\delta_l}{\sqrt{\sigma_l^2+\beta_l^2}} \quad \forall\ k=1,...,K-1,\ l=1,...,K. \qquad \text{(b.1)}$$

Therefore parameter ρ is not identifiable. Moreover, parameters σ_l^2 cannot be distinguished from β_l^2. Let us denote their sum by γ_l^2, where

$$\gamma_l = \sqrt{\sigma_l^2+\beta_l^2}.$$

There are $K(K-1)$ identifying functions (a.1), that we would like to use to identify the $(K-1)$ values of c_k, the K values of δ_l and the K values of γ_l, i.e. $3K-1$ unknowns. We follow Gagliardini, Gouriéroux (2015) and add the identifying constraints:

$$c_2 = 0, \quad \gamma_1 = 1.$$

Next, we proceed as follows:

(a) From (b.1) written for $k=2$, we identify $\frac{\delta_l}{\gamma_l}$. Given that $\gamma_1 = 1$, we get δ_1 identified.

(b) For $l = 1$, we have $\gamma_1 = 1$, hence we identify $c_k - \delta_1$, given (b.1). Therefore, all thresholds $c_k, \quad k = 2, ..., K$ are identified.

(c) Then the identifying functions can also be written as:

$$\frac{c_k - \delta_l}{\gamma_l} = \frac{c_k}{\gamma_l} - \frac{\delta_l}{\gamma_l}, k = 2, \ldots, K, l = 1, \ldots, K.$$

Therefore, from the identification of the ratios δ_l/γ_l result in (a), we identify all ratios c_k/γ_l. Then from the identification of the c_k's (b), we identify γ_l, $l = 1, ..., K$. Now, the c_k, γ_l are identified, and from (c), we identify δ_l, $l = 1, ..., K$.

B.2. Proof of Proposition 2

We have the following identifying functions of parameters:

$$(1) \quad \frac{c_k - \delta_l}{\sqrt{\sigma_l^2 + \beta_l^2(1-\rho^2)}}, k = 2, \ldots, K, l = 1, \ldots, K$$

$$(2) \quad \frac{\epsilon \beta_l \rho}{\sqrt{\sigma_l^2 + \beta_l^2(1-\rho^2)}}, l = 1, \ldots, K$$

$$(3) \quad \frac{c_k - \delta_l}{\sigma_l}, k = 2, \ldots, K, l = 1, \ldots, K$$

$$(4) \quad \frac{\epsilon \beta_l}{\sigma_l}, l = 1, \ldots, K.$$

Let us define:

$$\gamma_l = \sqrt{\sigma_l^2 + \beta_l^2(1-\rho^2)},$$

and use the identifying constraints:

$$\gamma_1 = 1, \quad c_2 = 0.$$

Then we proceed as follows:

(a) For $k = 2$, given $c_2 = 0$

and equation (1), we identify $\frac{\delta_l}{\gamma_l}$.

(b) For $k = 2$, given $c_2 = 0$,

and equation (3), we identify $\frac{\delta_l}{\sigma_l}, l = 1, \ldots, K$.

(c) Given that $\gamma_1 = 1$, it follows from (a) that parameter δ_1 is identified.

(d) Then, it follows from (b) that parameter σ_1 is identified.

(e) For $l = 1$ and equation (1), we identify

$$\frac{c_k - \delta_1}{\gamma_1} = c_k - \delta_1.$$

Hence, from (c), it follows that c_k, $k = 1, ..., K - 1$ are identified.

(f) From equation (1), the quantities

$$\frac{c_k}{\gamma_l} - \frac{\delta_l}{\gamma_l}$$

are identified since $\gamma_1 = 1$.

Then, by (a), the ratios $\frac{c_k}{\gamma_l}$ are identified.

(g) From (f) and (e), parameters γ_l, $l = 1, ..., K$ are identified.

(h) From (a) and (g), parameters δ_l, $l = 1, ..., K$ are identified.

(i) From (b) and (h), parameters σ_l, $l = 1, ..., K$ are identified.

(j) From equation (4) and result (i), parameters $\epsilon\beta_l$, $l = 1, ..., K$ are identified.

(k) From (2), we get the ratios $\frac{\epsilon\beta_l\rho}{\gamma_l}$ and given (g) we identify $\epsilon\beta_l\rho$, $l = 1, ..., K$

(l) Finally, from (j) and (k), we identify parameter ρ.

Appendix C: Proof of Uniform a.s. Convergence

Let us introduce a more precise notation: $\hat{p}_{k,l,t}(n, T) = (n_{k,l,t}/n_{l,t})$, where the indexes n, T are introduced to explicit the dependence in the number of individuals n and the number of dates T. Indeed, by the Hajek, Renyi inequality [Hajek, Renyi (1955), ?, ineq (2.8)], we get:

$$P[Max_{m \geq n} |\hat{p}_{k,l,t}(m, T) - p_{kl}(f_t, \theta_0)| > \epsilon] < \frac{1}{\epsilon^2} \sum_{m=n}^{\infty} \frac{1}{m^2} p_{kl}(f_t, \theta_0), \ \forall \epsilon > 0,$$

$$\iff P[Max_{m \geq n} |\hat{p}_{k,l,t}(m, T) - p_{kl}(f_t, \theta_0)| > \epsilon] < \frac{c}{\epsilon^2 n} p_{kl}(f_t, \theta_0), \ \forall \epsilon > 0,$$

where c is a constant. Then, it follows that:

$$P[Max_{t \leq T} Max_{m \geq n} |\hat{p}_{k,l,t}(m, T) - p_{kl}(f_t, \theta_0)| > \epsilon] < \frac{c}{\epsilon^2 n} \sum_{t=2}^{T} p_{kl}(f_t, \theta_0).$$

For n, T large, the upper bound: $\frac{c}{\epsilon^2}\frac{T}{n}\frac{1}{T}\sum_{t=2}^{T} p_{kl}(f_t, \theta)$ is equivalent to $\frac{c}{\epsilon^2}\frac{T}{n}E_0[p_{kl}(f_t, \theta_0)]$, by the geometric ergodicity of factor (f_t). Then by Assumption A5, we infer :

$$\lim_{T\to\infty} P[Max_{t\leq T} Max_{m\geq n}|\hat{p}_{k,l,t}(m, T) - p_{kl}(f_t, \theta_0)| > \epsilon] = 0,$$

and the required uniformity .

Therefore, after the normalization, the a.s. limit of the normalized composite log-likelihood is:

$$\begin{aligned}
&\lim_{n,T\to\infty} \text{ a.s. } \frac{1}{nT}\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{t=2}^{T}\left[n_{kl,t}\ log\ p_{kl}(\theta)\right] \\
=&\lim_{n,T\to\infty} \text{ a.s. } \frac{1}{nT}\sum_{k=1}^{K}\sum_{l=1}^{K}\sum_{t=2}^{T}\left[n_{l,t}\ \hat{p}_{kl,t}\ log\ p_{kl}(\theta)\right] \\
=&\lim_{T\to\infty} \text{ a.s. } \frac{1}{T}\sum_{l=1}^{K}\sum_{k=1}^{K}\sum_{t=2}^{T}\left[\lim_{n\to\infty}\left(\frac{n_{l,t}}{n}\ \hat{p}_{k,l,t}(n,T)\right)\ log\ p_{kl}(\theta)\right] \\
=&\lim_{T\to\infty} \text{ a.s. } \frac{1}{T}\sum_{l=1}^{K}\sum_{k=1}^{K}\sum_{t=2}^{T}\left[\mathbb{P}[c_k < y^*_{i,t} < c_{k+1}, c_l < y^*_{i,t-1} < c_{l+1}|f_t, f_{t-1}]\ log\ p_{kl}(\theta)\right] \\
&\text{(by the uniform a.s. convergence)} \\
=&\sum_{l=1}^{K}\sum_{k=1}^{K}\left[\lim_{T\to\infty} \text{ a.s.}\left[\frac{1}{T}\sum_{t=2}^{T}\mathbb{P}[c_k < y^*_{i,t} < c_{k+1}, c_l < y^*_{i,t-1} < c_{l+1}|f_t, f_{t-1}]\ log\ p_{kl}(\theta)\right]\right] \\
=&\sum_{l=1}^{K}\sum_{k=1}^{K}\left[\ \mathbb{P}[c_k < y^*_{i,t} < c_{k+1}, c_l < y^*_{i,t-1} < c_{l+1}]\ log\ p_{kl}(\theta)\right] \text{(since } f_t \text{ is geometrically ergodic)} \\
=&\sum_{l=1}^{K}\sum_{k=1}^{K}\left[\ p_l(\theta_0)\ p_{kl}(\theta_0)\ log\ p_{kl}(\theta)\right] \\
=&\sum_{l=1}^{K}\left[\ p_{l0}\ \left[\sum_{k=1}^{K} p_{kl}(\theta_0)\ log\ p_{kl}(\theta)\right]\right] \\
&\text{(where } p_{l0} \text{ is the true marginal stationary distribution of process } y^*) \\
\equiv& L_{cc,\infty}(c, \delta, \gamma).
\end{aligned}$$

More precisely, $L_{cc,n,T}(\theta)$ converge a.s. uniformly to

$$L_{cc,\infty}(c, \delta, \gamma) = \sum_{l=1}^{K}\left[\ p_{l0}\ \left[\sum_{k=1}^{K} p_{kl}(\theta_0)\ log\ p_{kl}(\theta)\right]\right].$$

Appendix D: Simulation Details and Additional Results

This section provides more details for the implementation of the simulation experiments and additional results.

D.1. Simulation Details

To compute the estimated asymptotic variances, we use the estimator $\hat{\Sigma}_\theta = \widehat{Var}\left(\sqrt{T}\left(\hat{\theta}_{nT} - \theta\right)\right)$ of the asymptotic covariance matrix $\Sigma_\theta = J_0^{-1}\left(\sum_{h=-\infty}^{\infty} I_{0h}\right) J_0^{-1}$ in Proposition 3, which is the heteroskedasticity and autocorrelation consistent (HAC) estimator

$$\hat{\Sigma}_\theta = \hat{J}_{n,T}^{-1}\left(\hat{I}_{0,n,T} + \sum_{h=1}^{T-1} k\left(\frac{h}{B_T}\right)\left(\hat{I}_{h,n,T} + \hat{I}'_{h,n,T}\right)\right)\hat{J}_{n,T}^{-1},$$

where

$$\hat{J}_{n,T} = -\frac{1}{T}\sum_{t=2}^{T}\left(\sum_{k=1}^{K}\sum_{l=1}^{K}\frac{n_{kl,t}}{n}\frac{\partial \log\left(p_{kl}\left(\hat{\theta}_{nT}\right)\right)}{\partial\theta\partial\theta'}\right),$$

$$\hat{I}_{h,n,T} = \frac{1}{T}\sum_{t=2}^{T-h}\left(\sum_{k=1}^{K}\sum_{l=1}^{K}\left[\frac{n_{kl,t}}{n} - \frac{1}{T}\sum_{t=2}^{T}\frac{n_{kl,t}}{n}\right]\frac{\partial \log\left(p_{kl}\left(\hat{\theta}_{nT}\right)\right)}{\partial\theta}\right) \times\left(\sum_{k=1}^{K}\sum_{l=1}^{K}\left[\frac{n_{kl,t+h}}{n} - \frac{1}{T}\sum_{t=2}^{T}\frac{n_{kl,t}}{n}\right]\frac{\partial \log\left(p_{kl}\left(\hat{\theta}_{nT}\right)\right)}{\partial\theta}\right)',$$

$k(\cdot)$ is a kernel function, and B_T is the bandwidth. The asymptotic inference is conducted using a quadratic spectral kernel with the bandwidth, which we set to $4\left(T/100\right)^{2/9}$, as suggested by Newey, West (1994). The kernel $k(\cdot)$ is a decreasing function, which accounts for the the decaying dependence between the observations at t and $t+h$ when h increases.

All derivatives were obtained using the numerical gradient function in Matlab. The second-order partial derivative in the definition of the variance estimator for the CL(2) is obtained using an outer-product argument. In particular, we note that:

$$J_0 = -\sum_{k=1}^{K}\sum_{l=1}^{K}\left(p_l(\theta_0)p_{kl}(\theta_0)\ \left(\frac{\partial^2 p_{kl}(\theta_0)}{\partial\theta\partial\theta'}\frac{1}{p_{kl}(\theta_0)} - \frac{\partial p_{kl}(\theta_0)}{\partial\theta}\frac{\partial p_{kl}(\theta_0)}{\partial\theta'}\frac{1}{(p_{kl}(\theta_0)^2)}\right)\right)$$

which is equivalent to

$$J_0 = -\sum_{l=1}^{K}\left(p_l(\theta_0)\ \frac{\partial^2 \sum_{k=1}^{K} p_{kl}(\theta_0)}{\partial\theta\partial\theta'}\right) + \sum_{k=1}^{K}\sum_{l=1}^{K}\left(p_l(\theta_0)p_{kl}(\theta_0)\ \frac{\partial p_{kl}(\theta_0)}{\partial\theta}\frac{\partial p_{kl}(\theta_0)}{\partial\theta'}\frac{1}{(p_{kl}(\theta_0)^2)}\right).$$

Since $\sum_{k=1}^{K} p_{kl}(\theta_0) = 1$, for any l,

$$\begin{aligned} J_0 &= \sum_{k=1}^{K}\sum_{l=1}^{K}\left(p_l(\theta_0)p_{kl}(\theta_0)\frac{\partial p_{kl}(\theta_0)}{\partial\theta}\frac{\partial p_{kl}(\theta_0)}{\partial\theta'}\frac{1}{(p_{kl}(\theta_0)^2)}\right) \\ &= \sum_{k=1}^{K}\sum_{l=1}^{K}\left(p_l(\theta_0)p_{kl}(\theta_0)\frac{\partial \log p_{kl}(\theta_0)}{\partial\theta}\frac{\partial \log p_{kl}(\theta_0)}{\partial\theta'}\right). \end{aligned}$$

Hence, we estimate J_0 using

$$\hat{J}_{n,T} = \sum_{k=1}^{K}\sum_{l=1}^{K}\left(\frac{n_{kl}}{n}\frac{\partial \log p_{kl}(\hat{\theta}_{nT})}{\partial\theta}\frac{\partial \log p_{kl}(\hat{\theta}_{nT})}{\partial\theta'}\right).$$

For the CL(2) estimators of variances, p_{kl}, n_{kl} and $n_{kl,t}$ are replaced with their lag 2 analog, respectively. In the different estimations, we replace β_1 by $\sqrt{\frac{1-\sigma^2}{1-\rho^2}}$ to comply with the identification condition $\gamma_1 = 1$ and the sign restriction on β_1. As explained before, we only need to impose the sign restriction on one of the β_l.

Given the treatment of the default state, the migrations from this state in the composite log-likelihood are taken into account using the estimated constant transition probabilities to states $k = 1, k = 2$ and $k = 3$. Furthermore, the composite log-likelihood at lag 2 and its derivative depend on an integral which cannot be computed analytically. This integral is the expected value of

$$\begin{aligned} \sum_{j=1}^{K}&\left[\Phi\left(\frac{c_{k+1}-\delta_j-\beta_j\rho f}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right)-\Phi\left(\frac{c_{k}-\delta_j-\beta_j\rho f}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right)\right] \\ &\times\left[\Phi\left(\frac{c_{j+1}-\delta_l-\beta_l f}{\sigma_l}\right)-\Phi\left(\frac{c_{j}-\delta_l-\beta_l f}{\sigma_l}\right)\right], \end{aligned}$$

where the source of the randomness is f, which follows a standard normal distribution.

Therefore, it can be approximated by

$$\frac{1}{S}\sum_{s=1}^{S}\sum_{j=1}^{K}\left[\left[\Phi\left(\frac{c_{k+1}-\delta_j-\beta_j\rho f_s}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right)-\Phi\left(\frac{c_k-\delta_j-\beta_j\rho f_m}{\sqrt{\sigma_j^2+\beta_j^2(1-\rho^2)}}\right)\right]\right.$$
$$\left.\times\left[\Phi\left(\frac{c_{j+1}-\delta_l-\beta_l f_s}{\sigma_l}\right)-\Phi\left(\frac{c_j-\delta_l-\beta_l f_s}{\sigma_l}\right)\right]\right],$$

where f_s is simulated $S = 1,000$ times from a normal distribution using $f_s = \rho f_{s-1} + \sqrt{1-\rho^2}\eta_s$, $f_0 \sim N(0,1)$ and $\eta_s \sim N(0,1)$, to incorporate the correlation among the factors.

D.2. Results for Design 1

Table 1: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 1 when $\rho = 0$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.10	0.07	0.05	0.11	0.08	0.06
$c_4 = 3.0$	0.26	0.17	0.13	0.26	0.19	0.14
$c_5 = 4.5$	0.45	0.31	0.22	0.43	0.32	0.23
$c_6 = 6.0$	0.69	0.49	0.35	0.64	0.47	0.34
$c_7 = 7.5$	0.98	0.69	0.49	0.89	0.64	0.46
$c_8 = 9.0$	1.30	0.94	0.66	1.20	0.84	0.60
Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.09	0.06	0.05	0.11	0.08	0.06
$\delta_2 = 1.0$	0.11	0.07	0.06	0.13	0.10	0.07
$\delta_3 = 2.5$	0.22	0.15	0.11	0.26	0.19	0.14
$\delta_4 = 4.0$	0.4	0.27	0.20	0.42	0.31	0.23
$\delta_5 = 5.5$	0.63	0.44	0.31	0.63	0.46	0.34
$\delta_6 = 7.0$	0.89	0.64	0.45	0.88	0.64	0.46
$\delta_7 = 8.5$	1.2	0.87	0.62	1.20	0.83	0.59
Unconditional Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\gamma_1 = 1.1$	0.07	0.05	0.03	0.08	0.05	0.04
$\gamma_2 = 1.1$	0.11	0.08	0.05	0.10	0.07	0.05
$\gamma_3 = 1.2$	0.15	0.11	0.08	0.13	0.09	0.07
$\gamma_4 = 1.2$	0.20	0.15	0.10	0.17	0.12	0.08
$\gamma_5 = 1.3$	0.25	0.19	0.13	0.21	0.15	0.10
$\gamma_6 = 1.3$	0.31	0.23	0.16	0.31	0.21	0.15

Table 2: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 1 when $\rho = 0.4$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.11	0.07	0.05	0.13	0.09	0.06
$c_4 = 3.0$	0.25	0.16	0.12	0.30	0.20	0.15
$c_5 = 4.5$	0.44	0.28	0.22	0.51	0.34	0.25
$c_6 = 6.0$	0.67	0.44	0.33	0.77	0.51	0.37
$c_7 = 7.5$	0.95	0.64	0.47	1.10	0.72	0.51
$c_8 = 9.0$	1.30	0.87	0.64	1.40	0.97	0.68

Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240\ell$
$\delta_1 = -0.5$	0.16	0.15	0.15	0.14	0.10	0.08
$\delta_2 = 1.0$	0.13	0.09	0.07	0.17	0.12	0.09
$\delta_3 = 2.5$	0.23	0.15	0.12	0.30	0.22	0.16
$\delta_4 = 4.0$	0.39	0.26	0.20	0.50	0.35	0.25
$\delta_5 = 5.5$	0.61	0.41	0.32	0.75	0.52	0.37
$\delta_6 = 7.0$	0.89	0.61	0.47	1.10	0.73	0.52
$\delta_7 = 8.5$	1.20	0.85	0.66	1.40	0.96	0.68

Unconditional Volatities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\gamma_1 = 1.1$	0.06	0.04	0.03	0.08	0.06	0.04
$\gamma_2 = 1.1$	0.09	0.07	0.05	0.120	0.08	0.06
$\gamma_3 = 1.2$	0.14	0.10	0.07	0.16	0.11	0.08
$\gamma_4 = 1.2$	0.19	0.14	0.10	0.22	0.14	0.10
$\gamma_5 = 1.3$	0.24	0.17	0.12	0.30	0.18	0.13
$\gamma_6 = 1.3$	0.30	0.21	0.15	0.50	0.26	0.18

Table 3: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 1 when $\rho = 0.7$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.13	0.08	0.07	0.21	0.13	0.07
$c_4 = 3.0$	0.29	0.19	0.15	0.52	0.32	0.16
$c_5 = 4.5$	0.50	0.33	0.25	0.93	0.58	0.28
$c_6 = 6.0$	0.76	0.50	0.38	1.40	0.92	0.43
$c_7 = 7.5$	1.10	0.72	0.53	2.00	1.30	0.61
$c_8 = 9.0$	1.50	0.98	0.72	2.40	1.70	0.79
Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.29	0.30	0.31	0.16	0.13	0.096
$\delta_2 = 1.0$	0.16	0.11	0.08	0.22	0.16	0.10
$\delta_3 = 2.5$	0.27	0.18	0.14	0.52	0.32	0.18
$\delta_4 = 4.0$	0.46	0.31	0.24	0.87	0.58	0.29
$\delta_5 = 5.5$	0.72	0.49	0.39	1.40	0.93	0.44
$\delta_6 = 7.0$	1.10	0.72	0.57	2.10	1.40	0.63
$\delta_7 = 8.5$	1.50	1.00	0.82	2.30	1.60	0.78
Unconditional Volatities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\gamma_1 = 1.1$	0.05	0.04	0.03	0.17	0.09	0.04
$\gamma_2 = 1.1$	0.10	0.07	0.05	0.24	0.15	0.06
$\gamma_3 = 1.2$	0.15	0.10	0.08	0.34	0.22	0.09
$\gamma_4 = 1.2$	0.20	0.14	0.10	0.43	0.29	0.12
$\gamma_5 = 1.3$	0.26	0.18	0.13	0.72	0.46	0.16
$\gamma_6 = 1.3$	0.31	0.22	0.15	0.49	0.35	0.20

Figure 1: Empirical PDF of t-Statistic for c_{k+1} when $\rho = 0$

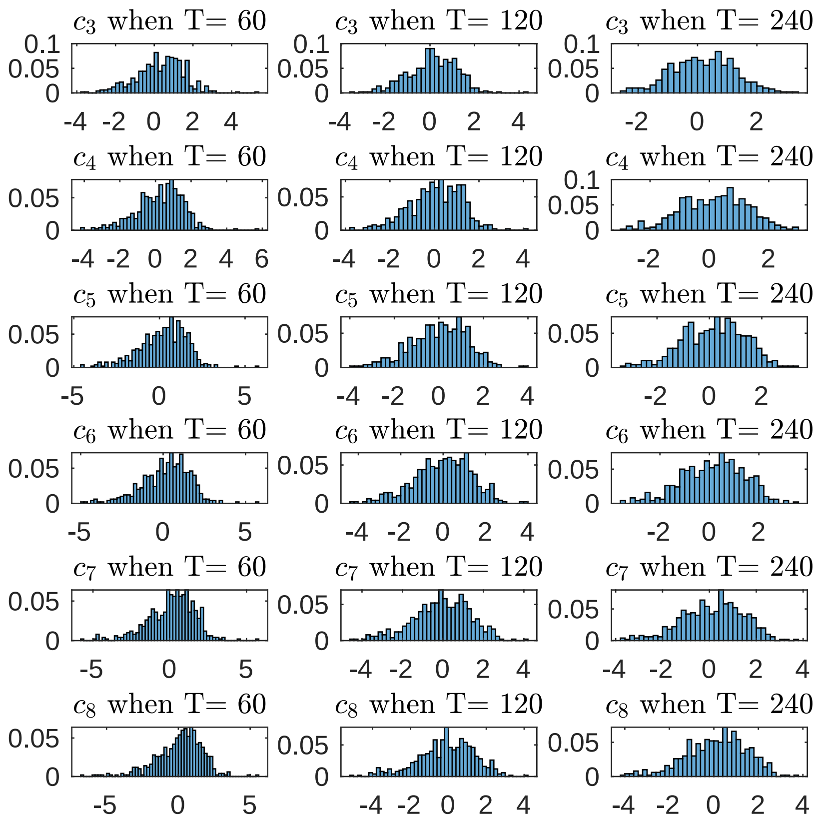


Figure 2: Empirical PDF of t-Statistic for c_{k+1} when $\rho = 0.4$

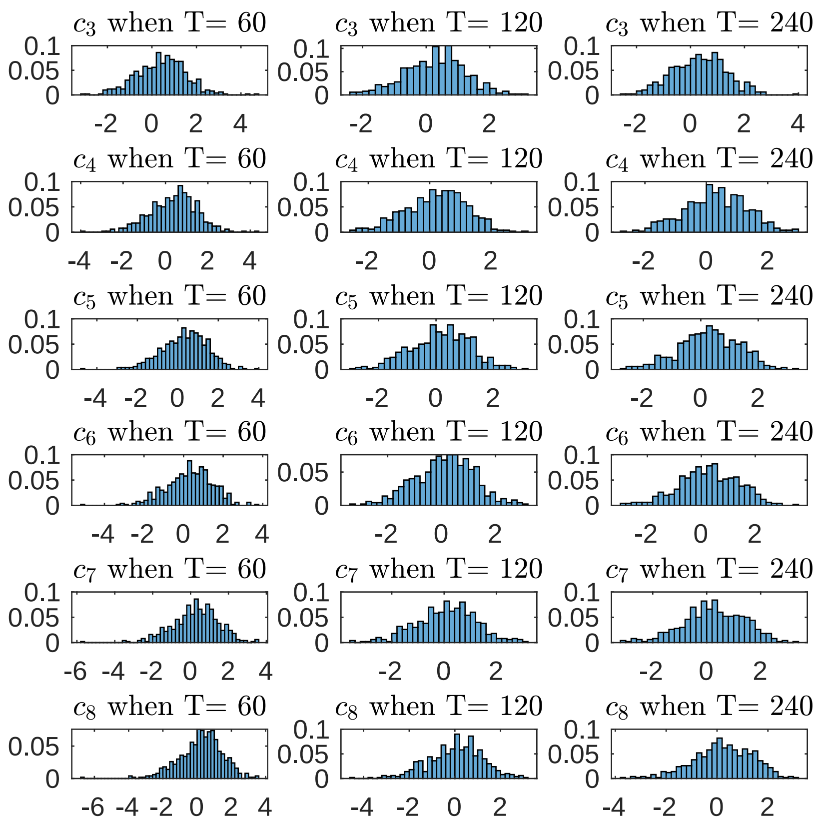


Figure 3: Empirical PDF of t-Statistic for c_{k+1} when $\rho = 0.7$

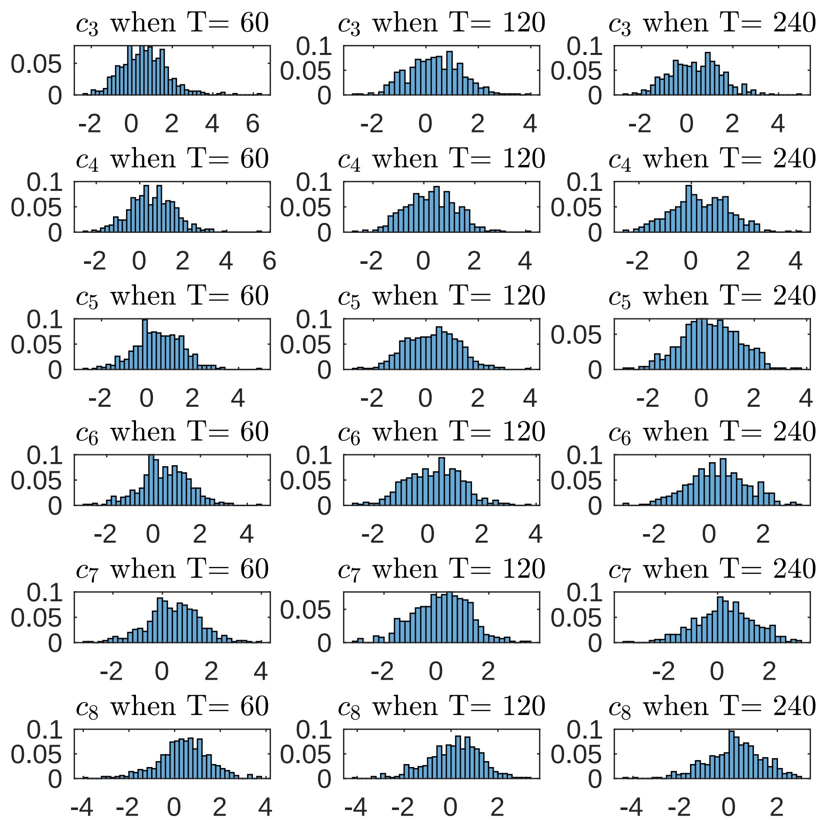


Figure 4: Empirical PDF of t Statistic for δ_l when $\rho = 0$

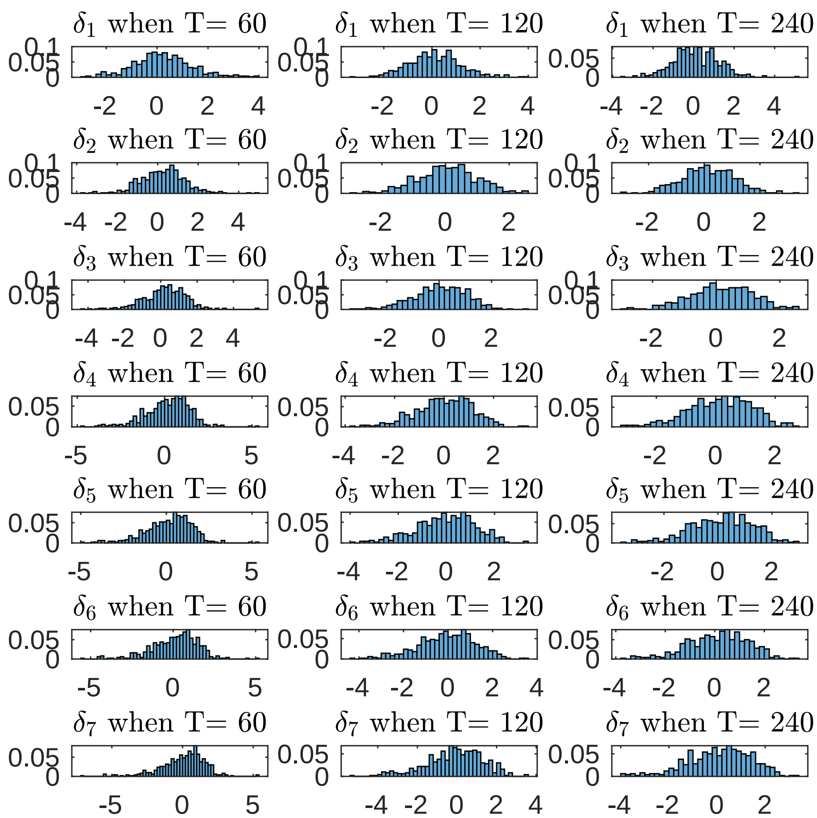


Figure 5: Empirical PDF of t-Statistic for δ_l when $\rho = 0.4$

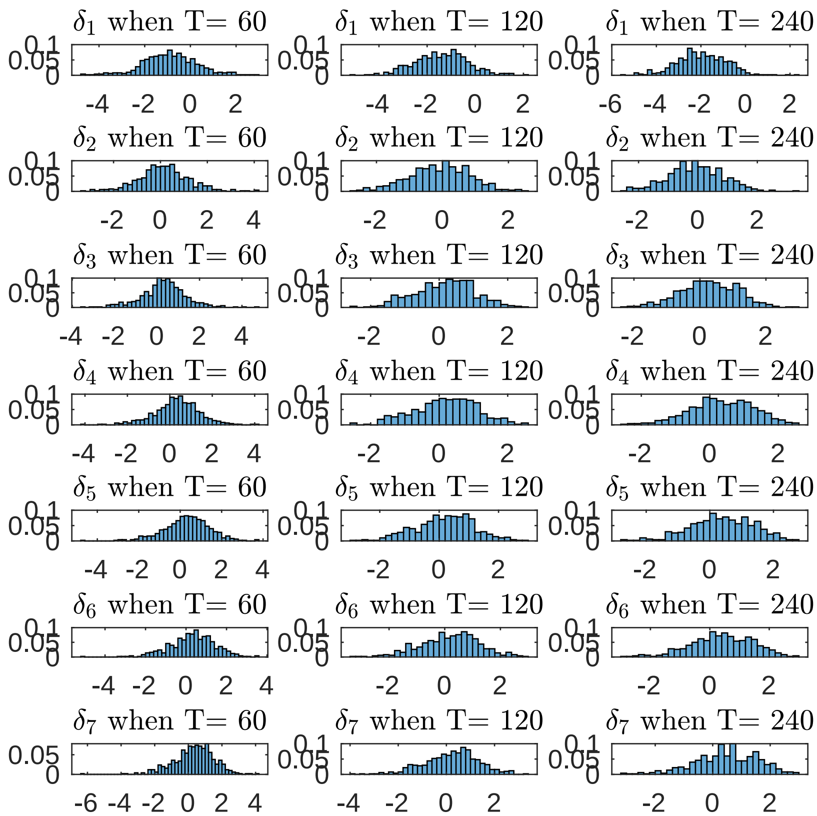


Figure 6: Empirical PDF of t-Statistic for δ_l when $\rho = 0.7$

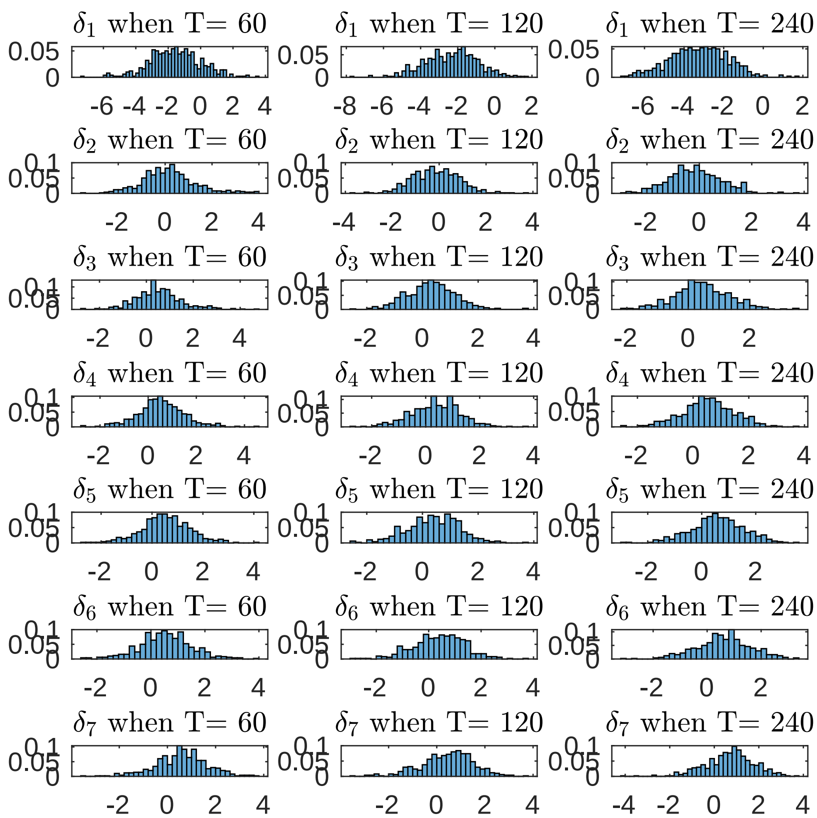


Figure 7: Empirical PDF of t-Statistic for γ_l when $\rho = 0$

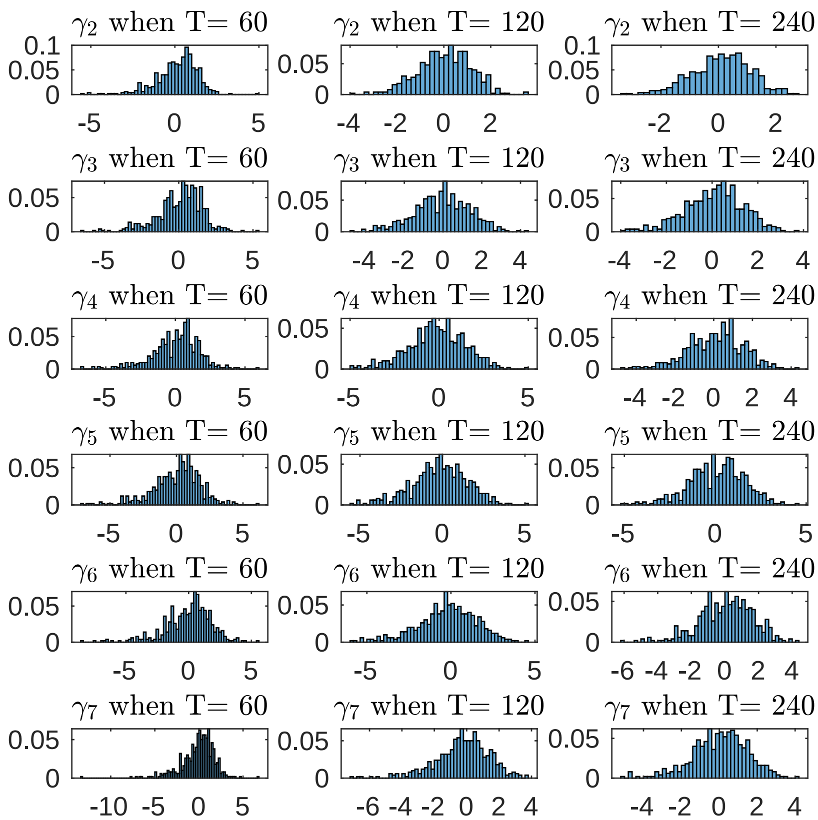


Figure 8: Empirical PDF of t-Statistic for γ_l when $\rho = 0.4$

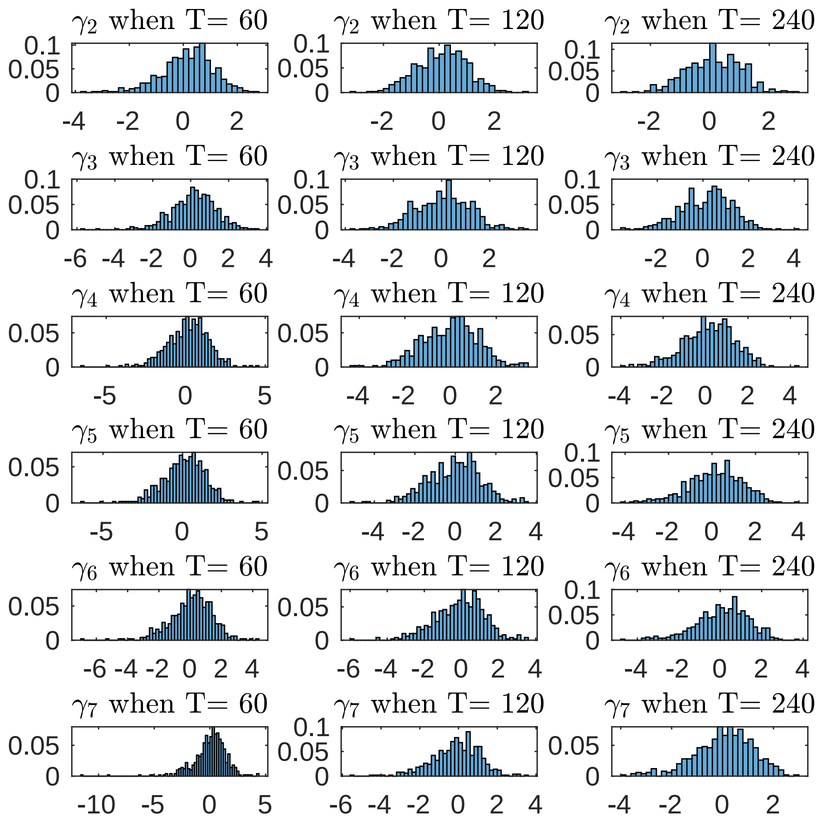
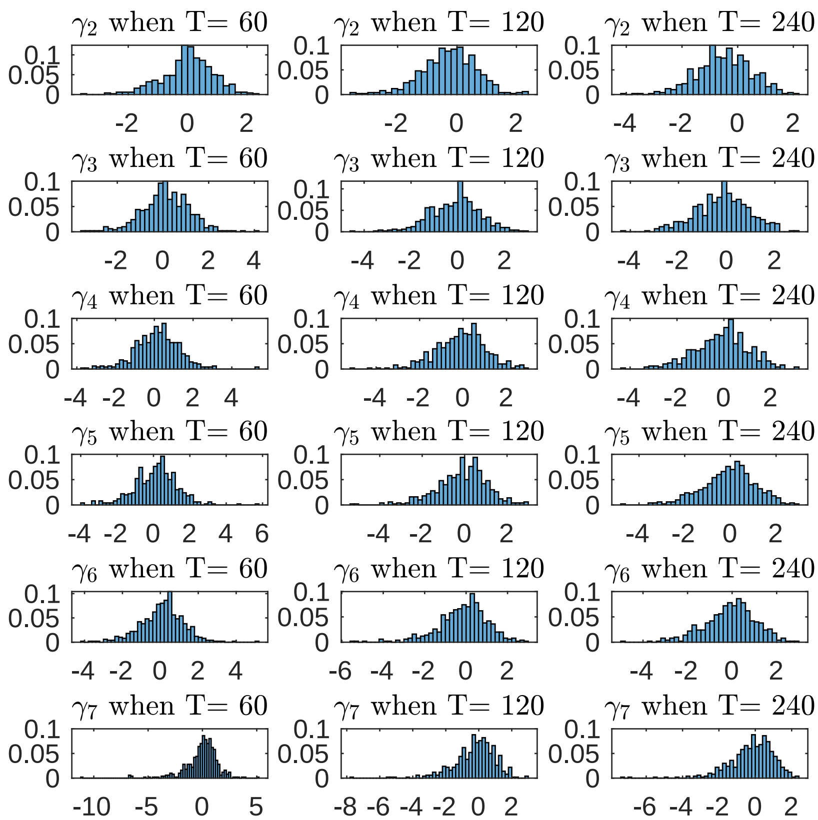


Figure 9: Empirical PDF of t-Statistic for γ_l when $\rho = 0.7$



D.3. Results for Design 2

Table 4: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 2 when $\rho = 0$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.09	0.07	0.05	0.18	0.11	0.08
$c_4 = 3.0$	0.15	0.11	0.08	0.33	0.21	0.14
$c_5 = 4.5$	0.16	0.11	0.08	0.50	0.32	0.23
$c_6 = 6.0$	0.16	0.12	0.091	0.72	0.50	0.36
$c_7 = 7.5$	0.25	0.20	0.16	1.00	0.74	0.55
$c_8 = 9.0$	0.45	0.35	0.30	1.40	1.10	0.79
Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.09	0.07	0.05	0.13	0.09	0.07
$\delta_2 = 1.0$	0.10	0.08	0.06	0.17	0.12	0.08
$\delta_3 = 2.5$	0.14	0.11	0.08	0.31	0.19	0.14
$\delta_4 = 4.0$	0.16	0.12	0.09	0.47	0.31	0.22
$\delta_5 = 5.5$	0.17	0.12	0.09	0.68	0.47	0.34
$\delta_6 = 7.0$	0.24	0.19	0.16	0.96	0.69	0.51
$\delta_7 = 8.5$	0.41	0.32	0.27	1.30	0.99	0.74
Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.74$	0.21	0.19	0.19	1.70	1.30	0.86
$\beta_3 = 0.78$	0.15	0.14	0.12	1.40	1.10	0.77
$\beta_4 = 0.82$	0.15	0.14	0.12	1.60	1.20	0.98
$\beta_5 = 0.86$	0.17	0.15	0.13	1.60	1.10	0.87
$\beta_6 = 0.9$	0.21	0.18	0.16	1.50	1.00	0.65
$\beta_7 = 0.95$	0.32	0.34	0.40	1.30	0.85	0.60
Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.71$	0.20	0.20	0.18	1.50	0.92	0.68
$\sigma_2 = 0.74$	0.13	0.12	0.13	1.30	0.93	0.62
$\sigma_3 = 0.78$	0.14	0.12	0.11	1.20	0.87	0.64
$\sigma_4 = 0.82$	0.16	0.15	0.12	1.30	1.00	0.82
$\sigma_5 = 0.86$	0.17	0.16	0.13	1.30	0.98	0.74
$\sigma_6 = 0.9$	0.24	0.20	0.20	1.30	0.84	0.58
$\sigma_7 = 0.95$	0.27	0.26	0.28	0.95	0.58	0.34
Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0$	$1.9e-3$	$1.9e-3$	$1.7e-3$	0.19	0.14	0.11

Table 5: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 2 when $\rho = 0.4$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.15	0.12	0.11	0.34	0.24	0.17
$c_4 = 3.0$	0.23	0.18	0.18	0.63	0.45	0.32
$c_5 = 4.5$	0.24	0.18	0.17	0.92	0.66	0.48
$c_6 = 6.0$	0.22	0.17	0.13	1.20	0.90	0.67
$c_7 = 7.5$	0.27	0.22	0.15	1.60	1.20	0.95
$c_8 = 9.0$	0.49	0.40	0.34	2.10	1.60	1.30

Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.13	0.11	0.09	0.17	0.12	0.09
$\delta_2 = 1.0$	0.14	0.11	0.09	0.30	0.22	0.16
$\delta_3 = 2.5$	0.21	0.17	0.15	0.59	0.44	0.31
$\delta_4 = 4.0$	0.23	0.18	0.16	0.89	0.65	0.46
$\delta_5 = 5.5$	0.23	0.17	0.14	1.20	0.89	0.66
$\delta_6 = 7.0$	0.28	0.23	0.17	1.60	1.20	0.93
$\delta_7 = 8.5$	0.52	0.43	0.38	2.10	1.60	1.30

Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.77$	0.18	0.16	0.15	0.96	0.67	0.45
$\beta_3 = 0.81$	0.15	0.12	0.10	0.87	0.64	0.42
$\beta_4 = 0.85$	0.16	0.13	0.10	0.93	0.62	0.45
$\beta_5 = 0.9$	0.19	0.15	0.12	0.99	0.67	0.42
$\beta_6 = 0.94$	0.23	0.20	0.16	1.10	0.82	0.44
$\beta_7 = 0.99$	0.33	0.29	0.27	1.10	0.82	0.50

Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.74$	0.18	0.17	0.16	1.30	0.79	0.56
$\sigma_2 = 0.77$	0.15	0.15	0.12	0.91	0.61	0.41
$\sigma_3 = 0.81$	0.17	0.15	0.11	0.82	0.63	0.43
$\sigma_4 = 0.85$	0.2	0.17	0.11	0.91	0.63	0.46
$\sigma_5 = 0.9$	0.25	0.23	0.13	0.88	0.70	0.46
$\sigma_6 = 0.94$	0.28	0.27	0.14	1.10	0.85	0.52
$\sigma_7 = 0.99$	0.45	0.46	0.23	1.10	0.73	0.55

Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0.4$	0.16	0.14	0.11	0.60	0.44	0.26

Table 6: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 2 when $\rho = 0.7$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.19	0.15	0.12	0.61	0.34	0.21
$c_4 = 3.0$	0.31	0.23	0.19	1.10	0.63	0.39
$c_5 = 4.5$	0.33	0.25	0.20	1.50	0.90	0.55
$c_6 = 6.0$	0.33	0.25	0.19	2.00	1.20	0.74
$c_7 = 7.5$	0.40	0.31	0.22	2.40	1.50	0.97
$c_8 = 9.0$	0.62	0.48	0.34	3.00	2.00	1.30

Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.25	0.21	0.20	0.23	0.17	0.13
$\delta_2 = 1.0$	0.19	0.15	0.11	0.39	0.27	0.18
$\delta_3 = 2.5$	0.28	0.24	0.20	0.94	0.60	0.38
$\delta_4 = 4.0$	0.31	0.27	0.23	1.40	0.90	0.57
$\delta_5 = 5.5$	0.34	0.29	0.27	1.90	1.20	0.75
$\delta_6 = 7.0$	0.47	0.41	0.37	2.40	1.50	0.98
$\delta_7 = 8.5$	0.81	0.71	0.63	3.00	2.00	1.30

Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.85$	0.23	0.19	0.18	0.86	0.60	0.46
$\beta_3 = 0.9$	0.19	0.15	0.14	0.88	0.62	0.46
$\beta_4 = 0.94$	0.18	0.16	0.13	0.95	0.65	0.49
$\beta_5 = 0.99$	0.23	0.20	0.18	0.99	0.67	0.49
$\beta_6 = 1.0$	0.29	0.25	0.20	1.20	0.86	0.52
$\beta_7 = 1.1$	0.37	0.32	0.26	1.20	0.87	0.49

Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.81$	0.21	0.20	0.23	1.40	1.10	1.00
$\sigma_2 = 0.85$	0.20	0.17	0.15	0.82	0.66	0.48
$\sigma_3 = 0.9$	0.19	0.17	0.15	0.86	0.69	0.51
$\sigma_4 = 0.94$	0.19	0.17	0.14	1.30	0.67	0.53
$\sigma_5 = 0.99$	0.21	0.19	0.17	1.00	0.82	0.56
$\sigma_6 = 1.0$	0.28	0.25	0.19	1.20	1.10	0.73
$\sigma_7 = 1.1$	0.40	0.33	0.29	1.30	1.10	0.75

Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0.7$	0.31	0.29	0.31	0.62	0.52	0.35

Table 7: Average of Estimated Downgrade Probabilities and Probabilities of Default (in %) for Design 2

$\rho = 0$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	32.52	33.29	33.48	33.55
$DP\,(2\|A)$	43.35	44.67	44.85	44.87
$PD\,(1\|A)$	$1.4e-7$	$1.4e-5$	$3.7e-6$	$1.5e-6$
$PD\,(12\|A)$	3.28	3.57	3.61	3.61
$PD\,(24\|A)$	3.19	3.16	3.19	3.19
$PD\,(36\|A)$	2.91	3.14	3.17	3.17

$\rho = 0.4$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	32.44	34.8	35.1	34.33
$DP\,(2\|A)$	44.05	45.89	46.37	46.33
$PD\,(1\|A)$	$8.2e-8$	$2.2e-4$	$6.3e-5$	$1.4e-5$
$PD\,(12\|A)$	3.37	3.81	3.88	3.93
$PD\,(24\|A)$	3.68	3.46	3.62	3.67
$PD\,(36\|A)$	3.16	3.44	3.61	3.66

$\rho = 0.7$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	34.7	37.69	38.21	38.48
$DP\,(2\|A)$	44.93	48.23	48.95	49.3
$PD\,(1\|A)$	$1.7e-5$	$2.0e-3$	$8.6e-4$	$3.4e-4$
$PD\,(12\|A)$	3.77	4.26	4.41	4.55
$PD\,(24\|A)$	4.53	4.03	4.18	4.33
$PD\,(36\|A)$	3.74	4.01	4.16	4.31

D.4. Results for Design 3

Table 8: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 3 when $\rho = 0$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.08	0.06	0.05	0.14	0.10	0.07
$c_4 = 3.0$	0.12	0.09	0.07	0.27	0.19	0.14
$c_5 = 4.5$	0.13	0.10	0.07	0.43	0.30	0.23
$c_6 = 6.0$	0.13	0.09	0.07	0.64	0.47	0.36
$c_7 = 7.5$	0.20	0.15	0.13	0.92	0.69	0.53
$c_8 = 9.0$	0.34	0.27	0.23	1.20	0.95	0.73
Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.09	0.07	0.05	0.12	0.09	0.06
$\delta_2 = 1.0$	0.10	0.07	0.06	0.15	0.10	0.07
$\delta_3 = 2.5$	0.13	0.09	0.07	0.25	0.18	0.13
$\delta_4 = 4.0$	0.14	0.10	0.07	0.4	0.28	0.21
$\delta_5 = 5.5$	0.14	0.10	0.08	0.60	0.43	0.33
$\delta_6 = 7.0$	0.19	0.15	0.13	0.85	0.63	0.48
$\delta_7 = 8.5$	0.31	0.24	0.21	1.20	0.88	0.68
Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.71$	0.21	0.18	0.19	1.60	1.10	0.80
$\beta_3 = 071$	0.13	0.11	0.11	1.20	0.85	0.65
$\beta_4 = 0.71$	0.14	0.12	0.11	1.30	0.96	0.73
$\beta_5 = 0.71$	0.14	0.12	0.12	1.10	0.87	0.62
$\beta_6 = 0.71$	0.16	0.14	0.13	0.90	0.68	0.45
$\beta_7 = 0.71$	0.24	0.26	0.30	0.87	0.56	0.38
Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.71$	0.19	0.19	0.19	1.20	0.85	0.61
$\sigma_2 = 0.74$	0.13	0.12	0.13	1.10	0.76	0.54
$\sigma_3 = 0.78$	0.12	0.10	0.09	0.91	0.67	0.50
$\sigma_4 = 0.82$	0.12	0.11	0.10	0.93	0.72	0.55
$\sigma_5 = 0.86$	0.14	0.11	0.10	0.82	0.63	0.46
$\sigma_6 = 0.9$	0.17	0.13	0.13	1.10	0.48	0.33
$\sigma_7 = 0.95$	0.19	0.18	0.19	0.52	0.31	0.21
Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0.0$	$1.7e-3$	$1.5e-3$	$1.5e-3$	0.17	0.13	0.09

Table 9: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 3 when $\rho = 0.4$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.14	0.11	0.09	0.32	0.21	0.15
$c_4 = 3.0$	0.2	0.15	0.13	0.59	0.40	0.27
$c_5 = 4.5$	0.20	0.15	0.12	0.85	0.59	0.40
$c_6 = 6.0$	0.18	0.13	0.09	1.20	0.82	0.56
$c_7 = 7.5$	0.22	0.18	0.13	1.50	1.10	0.77
$c_8 = 9.0$	0.40	0.33	0.26	2.00	1.40	1.00
Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.13	0.10	0.09	0.16	0.12	0.09
$\delta_2 = 1.0$	0.14	0.11	0.08	0.28	0.20	0.14
$\delta_3 = 2.5$	0.19	0.15	0.12	0.55	0.39	0.26
$\delta_4 = 4.0$	0.20	0.15	0.12	0.83	0.58	0.40
$\delta_5 = 5.5$	0.19	0.14	0.10	1.10	0.80	0.55
$\delta_6 = 7.0$	0.23	0.18	0.14	1.50	1.10	0.74
$\delta_7 = 8.5$	0.40	0.32	0.27	1.90	1.40	0.99
Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.74$	0.16	0.14	0.12	0.77	0.61	0.45
$\beta_3 = 074$	0.13	0.10	0.09	0.67	0.59	0.43
$\beta_4 = 0.74$	0.14	0.11	0.09	0.67	0.58	0.46
$\beta_5 = 0.74$	0.15	0.12	0.10	0.65	0.54	0.41
$\beta_6 = 0.74$	0.17	0.14	0.11	0.69	0.56	0.41
$\beta_7 = 0.74$	0.20	0.19	0.20	0.73	0.48	0.33
Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.74$	0.19	0.16	0.14	1.00	0.69	0.54
$\sigma_2 = 0.77$	0.13	0.11	0.09	0.64	0.52	0.40
$\sigma_3 = 0.81$	0.12	0.10	0.09	0.60	0.51	0.40
$\sigma_4 = 0.85$	0.13	0.11	0.09	0.59	0.48	0.38
$\sigma_5 = 0.9$	0.13	0.13	0.10	0.53	0.43	0.33
$\sigma_6 = 0.94$	0.16	0.15	0.12	0.55	0.45	0.35
$\sigma_7 = 0.99$	0.23	0.22	0.21	0.55	0.37	0.27
Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0.4$	0.14	0.12	0.12	0.57	0.43	0.34

Table 10: Average of Absolute Bias and Standard Errors for Estimated Parameters for Design 3 when $\rho = 0.7$

	Mean Absolute Bias			Standard Errors		
Thresholds	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$c_3 = 1.5$	0.17	0.13	0.11	0.48	0.28	0.19
$c_4 = 3.0$	0.26	0.21	0.17	0.89	0.54	0.36
$c_5 = 4.5$	0.28	0.22	0.17	1.30o	0.78	0.52
$c_6 = 6.0$	0.28	0.22	0.17	1.60	1.10	0.71
$c_7 = 7.5$	0.36	0.27	0.20	2.10	1.40	0.95
$c_8 = 9.0$	0.57	0.44	0.33	2.60	1.80	1.20

Intercepts	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\delta_1 = -0.5$	0.24	0.21	0.20	0.22	0.18	0.13
$\delta_2 = 1.0$	0.19	0.14	0.10	0.36	0.25	0.17
$\delta_3 = 2.5$	0.25	0.21	0.17	0.8	0.53	0.36
$\delta_4 = 4.0$	0.28	0.23	0.20	1.20	0.79	0.54
$\delta_5 = 5.5$	0.3	0.25	0.22	1.60	1.10	0.71
$\delta_6 = 7.0$	0.42	0.34	0.31	2.00	1.40	0.93
$\delta_7 = 8.5$	0.68	0.56	0.50	2.60	1.80	1.20

Factor Sensitivities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\beta_2 = 0.81$	0.22	0.18	0.16	0.86	0.62	0.56
$\beta_3 = 0.81$	0.17	0.14	0.12	0.86	0.68	0.59
$\beta_4 = 0.81$	0.18	0.14	0.13	0.89	0.65	0.54
$\beta_5 = 0.81$	0.23	0.18	0.17	0.95	0.76	0.54
$\beta_6 = 0.81$	0.23	0.19	0.18	0.98	0.80	0.53
$\beta_7 = 0.81$	0.28	0.22	0.18	0.78	0.60	0.38

Volatilities	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\sigma_1 = 0.81$	0.21	0.21	0.25	1.70	1.10	1.20
$\sigma_2 = 0.84$	0.18	0.14	0.14	0.81	0.61	0.56
$\sigma_3 = 0.90$	0.18	0.15	0.14	1.10	0.66	0.55
$\sigma_4 = 0.94$	0.18	0.14	0.13	0.75	0.60	0.49
$\sigma_5 = 0.99$	0.21	0.19	0.16	0.81	0.69	0.51
$\sigma_6 = 1.0$	0.23	0.21	0.19	0.81	0.78	0.56
$\sigma_7 = 1.1$	0.33	0.29	0.24	0.88	0.59	0.48

Autocorrelation	$T = 60$	$T = 120$	$T = 240$	$T = 60$	$T = 120$	$T = 240$
$\rho = 0.7$	0.32	0.31	0.31	0.61	0.56	0.40

Table 11: Average of Estimated Downgrade Probabilities and Probabilities of Default (in %) for Design 3

$\rho = 0$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	31.75	32.5	32.72	32.71
$DP\,(2\|A)$	43.27	44.53	44.74	44.67
$PD\,(1\|A)$	$2.5e-8$	$3.4e-6$	$1.0e-6$	$2.7e-7$
$PD\,(12\|A)$	3.26	3.57	3.58	3.58
$PD\,(24\|A)$	3.14	3.14	3.16	3.16
$PD\,(36\|A)$	2.91	3.12	3.14	3.13

$\rho = 0.4$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	32.45	33.94	34.25	34.26
$DP\,(2\|A)$	43.91	45.67	46.04	46.06
$PD\,(1\|A)$	$1.2e-7$	$3.5e-5$	$8.0e-6$	$2.6e-6$
$PD\,(12\|A)$	3.34	3.76	3.82	3.83
$PD\,(24\|A)$	3.58	3.35	3.43	3.48
$PD\,(36\|A)$	3.11	3.34	3.42	3.47

$\rho = 0.7$	True Value	$T = 60$	$T = 120$	$T = 240$
$DP\,(1\|A)$	34.01	36.85	37.31	37.5
$DP\,(2\|A)$	44.69	47.73	48.36	48.61
$PD\,(1\|A)$	$4.5e-6$	$7.4e-4$	$1.5e-4$	$9.3e-5$
$PD\,(12\|A)$	3.60	4.10	4.22	4.31
$PD\,(24\|A)$	4.337	3.82	3.89	3.90
$PD\,(36\|A)$	3.55	3.80	3.87	3.88

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