GCov-Based Portmanteau Test

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Abstract

We examine finite sample performance of the Generalized Covariance (GCov) residual-based specification test for semiparametric models with i.i.d. errors. The residual-based multivariate portmanteau test statistic follows asymptotically a χ^2 distribution when the model is estimated by the GCov estimator. The test is shown to perform well in application to the univariate mixed causal-noncausal MAR, double autoregressive (DAR) and multivariate Vector Autoregressive (VAR) models. We also introduce a bootstrap procedure that provides the limiting distribution of the test statistic when the specification test is applied to a model estimated by the maximum likelihood, or the approximate or quasi-maximum likelihood under a parametric assumption on the error distribution.

Keywords: Semi-Parametric Estimator, Generalized Covariance Estimator, Portmanteau Statistic, Causal-Noncausal Process, Double Autorregressive Process

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1 Introduction

We study the tests of the null hypothesis of absence of nonlinear serial dependence in time series. We show that the absence of nonlinear serial dependence in strictly stationary processes can be tested by a portmanteau test applied to their nonlinear transformations. This approach is applicable to either univariate or multivariate processes. It can also be extended to testing the residuals of a semi-parametric dynamic model for absence of nonlinear serial dependence, which leads to a specification test.

We examine the finite sample performance of the residual-based specification test for semi-parametric dynamic models with independent and identically distributed (i.i.d.) errors, when the model is estimated by the Generalized Covariance (GCov) estimator. Then, the residual-based multivariate portmanteau test statistic follows asymptotically a χ^2 distribution under the serial independence of errors [Gourieroux, Jasiak (2022)]. The test is shown to perform well in applications to the univariate mixed causal-noncausal MAR, double autoregressive (DAR) and multivariate Vector Autoregressive (VAR) models. We also introduce a bootstrap procedure that allows for using this specification test in models with i.i.d. errors when the model is estimated by a different method, such as the maximum likelihood or the approximate or quasi maximum likelihood method under parametric assumptions on the eerrror distribution.

The GCov test can be compared to the test of martingale difference condition of DeGrooijer (2023) which relies on a portmanteau test statistic based on the multivariate residuals and squared residuals. The GCov is more general in the sense that it is based on the autocovariances of nonlinear functions of the residuals, including possibly the residuals and their squares. An alternative approach for testing specification is based on the distance covariance. Davis and Wan (2022) consider the auto-distance covariance function and propose a specification test of the null hypothesis of residual independence. The advantage of the GCov-based test statistic, compared to their approach is that its asymptotic distribution is known, while the asymptotic distribution of the test statistic of Davis and Wan needs to be found by bootstrap. Chu (2023) also considers the null hypothesis of residual independence and uses the distance covariance approach. While the theoretical test statistic proposed by Chu has a known limiting distribution, in practice that theoretical test statistic needs to be approximated, which also requires bootstrapping the critical values.

While the GCov-based specification test is applicable to a variety of models with i.i.d. errors, it is particularly useful for testing the fit of causal-noncausal dynamic models with non-Gaussian errors. There is a growing interest in the univariate and multivariate mixed causal-noncausal models in the applied literature [Hecq, Lieb and Telg (2016), Hecq and Voisin (2021), Swensen (2022), Davis, Song (2020), Gourieroux, Jasiak (2017), (2022)]. The estimators available for this class of models are the semi-parametric GCov estimator [Gourieroux, Jasiak (2017), (2022)] and the Maximum Likelihood (ML) estimator [Davis, Song (2020)]. However, when the error distribution of a mixed model is misspecified, the ML estimator is inconsistent and the approach unreliable [Hecq, Lieb and Telg (2016)]. The advantage of the GCov estimator is that, unlike the maximum likelihood estimator, it does not require any distributional asssumptions on the errors. The semi-parametric GCov estimator is a one-step estimator, which is consistent, asymptotically normally distributed and semi-parametrically efficient.

The following notation will be used. For any $m \times n$ matrix A whose jth column is a_j, $j = 1, ..., n$, $vec(A)$ will denote the column vector of dimension mn defined as:

$$vec(A) = (a_1' ..., a_j', ..., a_n')',$$

where the prime denotes transposition. For any two matrices $A \equiv (a_{ij})$ and B, the Kronecker product $(A \otimes B)$ is the block matrix having $a_{ij}B$ for its (i, j)th block.

The paper is organized as follows. Section 2 discusses the tests of absence of linear and non-linear serial dependence in time series. Section 3 reviews the GCov estimator and presents the specification test based on the portmanteau test statistic and the GCov estimator. It also introduces the bootstrap procedure for specification testing when the test statistic is computed from an estimator different than the GCov, such as the (A)ML estimator. Section 4 presents the simulation results. Section 5 shows an empirical application to testing the specification of a dynamic model of non-fungible token (NFT) prices. Section 6 concludes. The technical results are given in Appendices A, B and C.

2 Tests of Absence of Linear and Non-Linear Serial Dependence in non-Gaussian Processes

This section examines the tests of the null hypothesis of absence of linear and nonlinear serial dependence in univariate or multivariate time series. It introduces a portmanteau test based on nonlinear transformations of the time series, which is applicable to strictly stationary time series with non-Gaussian marginal distributions and nonlinear dynamics.

The tests of absence of linear and nonlinear dependence considered in this paper concern technically the null hypothesis of zero values of autocovariances/autocorrelations of the series.

For processes with Gaussian distributions, the zero valued autocovariances are equivalent to serial independence, which becomes the null hypothesis of interest. Moreover, the asymptotic distribution of the test statistics of the independence hypothesis is determined under this null hypothesis. In the case of non-Gaussian processes, zero valued autocovariances do not imply serial independence. For this reason, instead of testing the null hypothesis of independence, we consider testing for the absence of (non)linear serial dependence. However, by analogy to the traditional literature, we use the independence hypothesis for deriving the asymptotic distributions of the test statistics.

2.1 Test of absence of linear serial dependence in univariate and multivariate time series

Let us recall the results that exist in the literature on the test of weak white noise hypothesis, i.e. on testing for the absence of linear dependence.

2.1.1 Univariate time series

Let us consider a univariate stationary time series (y_t) with finite fourth-order moments [3]. The test of the weak white noise hypothesis $H_0 = \{\gamma(h) = 0, h = 1, ..., H\}$, with $\gamma(h) = Cov(y_t, y_{t-h})$ is commonly based on the test statistic:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} \hat{\rho}(h)^2 = T \sum_{h=1}^{H} \frac{\hat{\gamma}(h)^2}{\hat{\gamma}(0)^2}, \tag{2.1}$$

where $\hat{\gamma}(h)$ and $\hat{\rho}(h)$ are the sample autocovariance and autocorrelation of order h, respectively, computed from a sample of T observations [4].

Under the null hypothesis of independence and standard regularity conditions, this statistic asymptotically follows a chi-square distribution $\chi^2(H)$ which H degrees of freedom [see Box, Pierce, 1970]. The following two subsections introduce the analogue of this statistic for stationary time series of higher dimension and tests of absence of nonlinear dependence.

2.1.2 Multivariate time series

Let us now consider a strictly stationary time series (Y_t) of dimension n with finite fourth-order moments. The null hypothesis is $H_0 = \{\Gamma(h) = 0, h = 1, ..., H\}$, where $\Gamma(h) =$

[3] The existence of moments up to order 4 is needed for deriving the asymptotic variance of $\hat{\gamma}(h)$ under the asymptotic normality.

[4] In this Section the index T of the estimators, e.g. $\hat{\gamma}_T(h)$ is omitted to simplify the notation.

$Cov(Y_t, Y_{t-h})$ is the autocovariance of order h. The multivariate equivalent of the test statistic (2.1) is:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} Tr[\hat{R}^2(h)], \tag{2.2}$$

where $\hat{R}^2(h)$ is the sample analogue of the multivariate R-square defined by:

$$R^2(h) = \Gamma(h)\Gamma(0)^{-1}\Gamma(h)'\Gamma(0)^{-1}. \tag{2.3}$$

Since

$$\hat{R}^2(h) = \hat{\Gamma}(0)^{1/2}[\hat{\Gamma}(0)^{-1/2}\hat{\Gamma}(h)\hat{\Gamma}(0)^{-1}\hat{\Gamma}(h)'\hat{\Gamma}(0)^{-1/2}]\hat{\Gamma}(0)^{-1/2}, \tag{2.4}$$

this matrix is equivalent up to a change of basis to the matrix within the brackets, which is symmetric and positive-definite. Therefore, it is diagonalisable, with a trace equal to the sum of its eigenvalues, which are the squares of the canonical correlations between Y_t and Y_{t-h}, denoted by $\hat{\rho}_i^2(h), i = 1, ..., n$ [Hotelling (1936)]. Therefore:

$$\hat{\xi}_T(H) = T \sum_{h=1}^{H} Tr[\hat{\Gamma}(h)\hat{\Gamma}(0)^{-1}\hat{\Gamma}(h)'\hat{\Gamma}(0)^{-1}] = T \sum_{h=1}^{H}[\sum_{i=1}^{n} \hat{\rho}_i(h)^2].$$

Under the null hypothesis of strong white noise, this statistic follows asymptotically a chi-square distribution $\chi^2(nH)$ [see, e.g. Robinson (1973), Anderson (1999), Section 7, Anderson (2002), Section 5].

There exist alternative test statistics that have been introduced in the literature and are asymptotically equivalent to the test statistic $\hat{\xi}_T(H)$ under the null hypothesis. For example, we can consider the Seemingly Unrelated Regression (SUR) model:

$$Y_t = \alpha + B_1 Y_{t-1} + \cdots + B_H Y_{t-H} + u_t, \tag{2.5}$$

and introduce the statistic by using Frisch-Waugh-Lovell theorem:

$$\tilde{\xi}_{1T}(H) = T\ Tr[\hat{\Gamma}^*(1)\hat{\Gamma}^*(0)^{-1}\hat{\Gamma}^*(1)'\hat{\Gamma}^*(0)^{-1}], \tag{2.6}$$

where $\Gamma^*(1) = Cov(Y_t, \underline{Y_{t-1}}), \Gamma^*(0) = V(\underline{Y_{t-1}})$ and $\underline{Y_{t-1}} = (Y'_{t-1}, ..., Y'_{t-H})'$.

Under the null hypothesis of strong white noise, the explanatory variables in (2.6) are (asymptotically) uncorrelated, which explains the possibility of replacing the canonical correlation analysis of dimension nH by H canonical correlations of dimension n only.

The statistics $\hat{\xi}_T(H)_{1T}$ and $\hat{\xi}_T(H)$ are asymptotically equivalent, that is:

$$\hat{\xi}_{1,T}(H) - \hat{\xi}_T(H) = o_p(1).$$

Then, these test statistics have the same asymptotic distribution under the serial independence condition. This result is demonstrated in Appendix A.

2.2 Tests of absence of nonlinear serial dependence in time series

Let us extend these results and consider a strictly stationary time series y_t of dimension n. The results of Section 2.1 suggest that the null hypothesis of absence of nonlinear and linear dependence in univariate or multivariate time series y_t can be tested by transforming it into a vector of nonlinear transforms, to which the test statistic $\hat{\xi}_T(H)$ can be applied [see Chan et al. (2006), Gourieroux, Jasiak (2022)].

More specifically, we compute the test statistics $\hat{\xi}_T(H)_a$ from nonlinear transforms Y_t^a of a univariate or multivariate non-Gaussian time series y_t. Let us consider a vector of nonlinear functions of a univariate strictly stationary process y_t. The process is transformed into a system of higher dimension by considering nonlinear differentiable functions of y_t, such as the squares, absolute values, or logarithms, for example. In particular if (y_t) has no finite fourth-order moment, then it can be replaced by a transformed multivariate process Y_t^a with a finite fourth-order moment.

Let us introduce K nonlinear functions $a_1, ..., a_K$ of the process that transform it into a multivariate process of dimension K with components $a_k(y_t)$:

$$Y_t^a = \begin{pmatrix} a_1(y_t) \\ \vdots \\ a_K(y_t) \end{pmatrix},$$

where $a_1(y_t) = y_t$ is the time series itself, to capture the linear dependence. We compute the sample autocovariances of the transformations $a_k(y_t), k = 1, ..., K$:

$$\hat{\Gamma}^a(h) = \frac{1}{T}\sum_{t=h}^{T} Y_t^a Y_{t-h}^{a\prime} - \frac{1}{T}\sum_{t=h}^{T-1} Y_t^{a\prime} \frac{1}{T}\sum_{t=h+1}^{T} Y_{t-h}^a.$$

We assume that $a_k(y_t),\ k = 1, ..., K$ have finite variances. Once a set of transformations is determined, the null hypothesis becomes:

$$H_{0,a} = (\Gamma^a(h) = 0,\ h = 1, ..., H).$$

In general, the test statistic:

$$\hat{\xi}_T(H)_a = T \sum_{h=1}^{H} Tr \hat{R}_a^2(h) \tag{2.7}$$

can be either computed from a transformed univariate or multivariate process. In each case the dimension of the process is increased. If the combined dimension of the process is K, then the test statistic (2.11) follows asymptotically a $\chi^2(K^2 H)$ distribution [see, e.g. Robinson (1973), Chitturi (1976), Anderson (1999), Section 7, Anderson (2002), Section 5].

3 Goodness of Fit Test

The test statistics introduced in the previous Section can be used for testing the goodness of fit of semi-parametric nonlinear models for strictly stationary time series with i.i.d. errors and parameter vector θ describing their dynamics. In this context, $\hat{\xi}_T(H)$ is computed from a vector of univariate series of residuals and its nonlinear transforms, instead of an observed time series, which changes its limiting distribution to $\chi^2(K^2 H - dim(\theta))$ [Gourieroux, Jasiak (2022)].

3.1 The Semi-Parametric Model

Let us consider a strictly stationary process (Y_t) satisfying a semi-parametric model studied in Gourieroux, Jasiak (2022):

$$g(\tilde{Y}_t; \theta) = u_t, \tag{3.1}$$

where g is a known function satisfying the regularity conditions, where $dim(g) = dim(u_t) = J$, $\tilde{Y}_t = (Y_t, Y_{t-1}, \ldots, Y_{t-p})$, p is a non-negative integer, (u_t) is an i.i.d. sequence (not necessarily with mean zero) with a common marginal density function f and θ is an unknown parameter vector of dimension denoted by $dim(\theta)$. We assume that the model is well-specified, the true value of parameter θ is θ_0 and the true error density is f_0. Model (3.1) does not imply a nonlinear causal autoregressive specification of order p for process (Y_t) because the dimension of Y_t can be strictly larger (resp. smaller) than the dimension of u_t. Hence, model (3.1) is not directly invertible with respect to Y_t. Moreover, u_t is not assumed to be independent of $\tilde{Y}_{t-1}$. Therefore, the information generated by the current and lagged values of Y_t does not necessarily coincide with the information generated by the current and lagged values of u_t. The errors u_t are not necessarily interpretable as either the causal, or non-causal innovations.

Examples of semi-parametric nonlinear models are given below:

Example 1: Double Autoregressive and Stochastic Volatility Models

i) A model encompassing the Double-Autoregressive (DAR) model [Ling (2007)] can be written as:

$$y_t = \phi y_{t-1} + u_t \sqrt{w + \alpha y_{t-1}^2},$$

where $w > 0, \phi \geq 0, \alpha \geq 0$, $\theta = (w, \phi.\alpha)'$ and the u_t's are i.i.d. with distribution f. We assume that the regularity conditions on functions θ, f ensuring the existence of a stationary solution are satisfied. Moreover, the initial value y_0, is assumed drawn in the stationary distribution. Then

$$g(\tilde{Y}_t; \theta) = [(y_t - \phi y_{t-1})/\sqrt{w + \alpha y_{t-1}^2} = u_t,$$

is the semi-parametric representation (3.1) of this model.

Example 2: MAR(r,s) Model

The mixed noncausal autoregressive MAR(r,s) process is defined as:

$$(1 - \phi_1 L - \phi_2 L^2 - ... - \phi_r L^r)(1 - \psi_1 L^{-1} - \psi_2 L^{-2} - ... - \psi_s L^{-s}) y_t = u_t, \tag{3.2}$$

where the errors are independent, identically distributed and such that $E(|u_t|^\delta) < \infty$ for $\delta > 0$ and Φ and Ψ are two polynomials of degrees r and s, respectively, with roots strictly outside the unit circle and such that $\Phi(0) = \Psi(0) = 1$. In particular,the MAR(1,1) model:

$$(1 - \phi L)(1 - \psi L^{-1}) y_t = u_t, \tag{3.3}$$

where the errors are independent, identically distributed and such that $E(|u_t|^\delta) < \infty$ for $\delta > 0$ and parameters ϕ and ψ are two autoregressive cofficients that are strictly less than one. Coefficient ϕ represents the standard causal persistence while coefficient ψ depicts noncausal persistence. In this case: $\theta = (\phi, \psi)'$, $p = 2 = dim(\theta)$ and

$$g(\tilde{Y}_t; \theta) = \Phi(L)\Psi(L^{-1}) y_t = u_t$$

is the semi-parametric representation (3.1) of this model.

Example 3: Causal-Noncausal VAR Model

The multivariate causal-noncausal VAR(p) process is defined by:

$$Y_t = \Phi_1 Y_{t-1} + \cdots + \Phi_p Y_{t-p} + u_t,$$

where $\theta = [vec\Phi_1', ..., vec\Phi_p']'$ and error u_t is a multivariate non-Gaussian i.i.d. process with finite fourth order moments. We assume that the roots of the characteristic equation $det(Id - \Phi_1 \lambda - \cdots \Phi_p \lambda^p) = 0$ are of modulus either strictly greater, or smaller than one. Then, there exists a unique (strictly) stationary solution (Y_t) with a two-sided $MA(\infty)$ representation, which satisfies model (3.1) with:

$$g_t(\tilde{Y}_t, \theta) = Y_t - \Phi_1 Y_{t-1} - \cdots - \Phi_p Y_{t-p} = u_t.$$

The causal-noncausal VAR(p) model has been studied in Gourieroux, Jasiak (2017),(2022) and Davis, Song (2020). The error u_t cannot be interpreted as an innovation. Moreover, even if function g is linear in the current and lagged values of Y_t, the assumption of strict stationarity of Y_t implies a nonlinear causal dynamics of Y_t with predictions $E(Y_t|\underline{Y_{t-1}})$ nonlinear in $\underline{Y_{t-1}} = (Y_t, Y_{t-1}, ...)$ and past-conditional heteroscedasticity $V(Y_t|\underline{Y_{t-1}})$.

Model (3.1) can be transformed into a system of higher dimension by considering nonlinear transformations of u_t. Let us introduce K nonlinear transformations $a_1, ..., a_K$. Then we have:

$$\begin{aligned} a_k[g(\tilde{y}_t; \theta)] &= a_k(u_t), \ k = 1, ..., K, \\ \text{or, equivalently } \ a[g(\tilde{y}_t; \theta)] &= a(u_t) = v_t, \end{aligned} \tag{3.4}$$

where the transformed process (v_t) is also an i.i.d. process. From now on, the subscripts of transformations a are disregarded for clarity.

3.2 The GCov Test

The GCov estimator is a one-step estimator introduced in Gourieroux, Jasiak (2022). Suppose that we observe $Y_1, ..., Y_T$. The GCov estimator of the parameter θ of model dynamics is:

$$\hat{\theta}_T(H) = Argmin_\theta \sum_{h=1}^{H} Tr[\hat{R}_T^2(h, \theta)], \tag{3.5}$$

where:

$$\hat{R}_T^2(h, \theta) = \hat{\Gamma}_T(h; \theta)\hat{\Gamma}_T(0; \theta)^{-1}\hat{\Gamma}_T(h; \theta)'\hat{\Gamma}_T(0; \theta)^{-1}, \tag{3.6}$$

and $\hat{\Gamma}_T(h; \theta)$ is covariance function between $g(\tilde{Y}_t; \theta)$ and $g(\tilde{Y}_{t-h}; \theta)$ of dimension K. The sample autocovariances of $g(\tilde{Y}_t; \theta)$ are computed from $t = p + H + 1$ up to T. These sample autocovariances have to be divided by T instead of $(T - H - p)$ to ensure that the sequence of multivariate sample autocovariances remains positive semi-definite.

Let us assume that model (3.1) is well-specified. Then, the GCov estimator is consistent, asymptotically normally distributed and attains a semi-parametric efficiency bound, under standard regularity conditions [Gourieroux, Jasiak (2022)].

The GCov estimator is used to build a test of model specification, which test the null hypothesis of absence of linear or nonlinear serial dependence in the (nonlinear transformations of) residuals of the model. The test statistic for testing the model specification is the portmanteau test statistic evaluated at $\hat{\theta}_T$ [Gourieroux, Jasiak (2022)]:

$$\hat{\xi}_T(H) = TL_T(\hat{\theta}_T), \tag{3.7}$$

where:

$$L_T(\hat{\theta}_T) = \sum_{h=1}^{H} Tr[\hat{R}_T^2(h, \hat{\theta}_T)],$$

and the estimated autocovariances $\hat{\Gamma}_T(h; \hat{\theta}_T)$ are the sample autocovariances of the residuals $\hat{u}_{t,T} = u_t(\hat{\theta}_T) = g(\bar{Y}_t, \hat{\theta}_T)$.

Under the null hypothesis of absence of linear or nonlinear serial dependence, $\hat{\xi}_T(H)$ follows asymptotically the chi-square distribution with degrees of freedom equal to $K^2 H - dim\theta$ [see, Gourieroux, Jasiak (2022), Proposition 4]. This result holds only for the GCov estimator $\hat{\theta}_T$.

3.3 Asymptotic Properties of the GCov Test

This Section describes the asymptotic size and local power of the GCov test.

3.3.1 Null Hypothesis and Asymptotic Size

Let us now clarify the definition of the null hypothesis in this semi-parametric framework. As mentioned earlier there are two types of parameters: vector θ defining the dynamics and functional parameter f defining the error distribution. Then, the theoretical autocovariances $\Gamma(h; \theta, f)$ are functions of both θ and f. The null hypothesis becomes:

$$H_0 : \{\Gamma_0(h) = 0, \ h = 1, ..., H\},$$

where $\Gamma_0(h)$ is the true autocovariance. In terms of parameters θ, f, it corresponds to $\Theta_0 = \{\theta, f : (\Gamma(h; \theta, f) = 0\}$.

So far, the transformation a has not been introduced to simplify the notation. When these transformations are accounted for, the null hypothesis becomes:

$$H_{0,a} \approx \Theta_{0,a} = \{\theta, f : \Gamma_{0,a}(h; \theta, f) = 0, \ h = 1, ..., H\}.$$

When the error terms u_t are serially independent, the nonlinear autocovariances $\Gamma_0(h) = 0, \ \forall h$. Hence, if the model is correctly specified and $H \geq p$, the GCov test of the null hypothesis H_0 tests the absence of linear and nonlinear dependence in the errors u_t. Because one cannot consider all the lags and nonlinear transformations, the null hypothesis of absence of (non)linear dependence is not equivalent to the independence condition, but is arbitrarily close to it, depending on lag H and the number of nonlinear transformations considered.

The test of the hypothesis H_0 at level α is performed as follows: the null hypothesis H_0 is rejected when $\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H - dim\theta)$ and H_0 is not rejected otherwise. The asymptotic size tends to the nominal size when $T \to \infty$:

$$\lim_{T\to\infty} P_{\theta,f}[\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H - dim\theta)] = \alpha,\ \forall\theta, f \in \Theta_0,\ \forall\alpha.$$

3.3.2 Local Alternatives and Local Power

Because the semi-parametric model depends not only on the vector of dynamic parameters θ, but also on the functional parameter f, the general alternative hypothesis is difficult to formulate. In particular, the alternative could contain models in which the u_t's are serially dependent. Then, the parametrization would have to be enlarged to accommodate these cases. From now on the transformation a subscripts are disregarded for clarity.

We assume a parametrized (fixed) alternative defined by:

$$H_1^* = \{g^*(\tilde{Y}_t; \theta, \gamma) = u_t,\ \text{where } u_t \text{ are i.i.d.,}\} \tag{3.8}$$

with an additional (scalar) parameter γ, a known function g^*, such that $g^*(\tilde{Y}_t; \theta, 0) = g(\tilde{Y}_t; \theta)$ and the dimension $dim(g^*) = dim(u_t)$. Under H_1^*, the autocovariances depend on θ, γ and the marginal distribution f of errors u_t. These autocovariances are denoted by $\Gamma(h; \theta, \gamma, f)$.

The alternative hypothesis is parametrized by θ, γ, f. In terms of the autocovariances, the (fixed) alternative hypothesis is:

$$H_1 = \{\Gamma(h; \theta, \gamma, f) = 0,\ h = 1, ..., H\}$$

and the null hypothesis is:

$$H_0 = \{\Gamma(h; \theta, 0, f) = 0,\ h = 1, ..., H\} = \{\gamma = 0\}$$

Example DAR(1) model The (fixed) alternatives are for example:

1. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t\sqrt{w + \alpha y_{t-1}^2}$
and $H_{11}^* : u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y_{t-1}^2} = g^*(\tilde{Y}_t; \theta, \gamma)$
2. $y_t = \phi_1 y_{t-1} + u_t\sqrt{w + \alpha y_{t-1}^2 + \gamma y_{t-2}^2}$

and $H^*_{12}: u_t = (y_t - \phi_1 y_{t-1})/\sqrt{w + \alpha y^2_{t-1} + \gamma y^2_{t-2}} = g^*(\tilde{Y}_t; \theta, \gamma)$

3. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t \sqrt{w + \alpha y^2_{t-1} + \gamma y^2_{t-2}}$

and $H^*_{13}: u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y^2_{t-1} + \gamma y^2_{t-2}} = g^*(\tilde{Y}_t; \theta, \gamma)$

4. $y_t = \phi_1 y_{t-1} + (u_t + \gamma u_{t-1})\sqrt{w + \alpha y^2_{t-1}}$

and $H^*_{14}: u_t = (\frac{1}{1-\gamma L})\left[(y_t - \phi_1 y_{t-1})/\sqrt{w + \alpha y^2_{t-1}}\right]$ for $\gamma \neq 1$. We see that $u_t \neq g^*(\tilde{Y}_t; \theta, \gamma)$ with $\tilde{Y}_t = Y_{t-1}, ..., Y_{t-p}$ where p is a fixed lag. Therefore, we will not be able to test against this alternative hypothesis.

The local alternatives are defined in a neighbourhood of the true value θ_0, f_0 satisfying the null hypothesis. We consider parametric directional alternatives, where:

$$\theta_T \approx \theta_0 + \mu/\sqrt{T}, \quad \gamma_T \approx \nu/\sqrt{T}, \quad f_T \approx f_0.$$

Example DAR(1) model The local alternatives of the DAR model obtained by Taylor expansion about $\gamma = 0$ for (fixed) hypotheses 1 to 3 given above are:

1. $y_t = \phi_1 y_{t-1} + \gamma y_{t-2} + u_t \sqrt{w + \alpha y^2_{t-1}}$

and $H^*_{L11}: u_t = (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\sqrt{w + \alpha y^2_{t-1}} = g^*(\tilde{Y}_t; \theta, \gamma)$

2. $y_t \approx \phi_1 y_{t-1} + u_t \left[\sqrt{w + \alpha y^2_{t-1}} + \frac{\gamma}{2}\frac{y^2_{t-2}}{\sqrt{w+\alpha y^2_{t-1}}}\right]$

and $H^*_{L12}: u_t \approx (y_t - \phi_1 y_{t-1})/\left[\sqrt{w + \alpha y^2_{t-1}} + \frac{\gamma}{2}\frac{y^2_{t-2}}{\sqrt{w+\alpha y^2_{t-1}}}\right] \approx g^*(\tilde{Y}_t; \theta, \gamma)$

3. $y_t \approx \phi_1 y_{t-1} + \gamma y_{t-2} + u_t \left[\sqrt{w + \alpha y^2_{t-1}} + \frac{\gamma}{2}\frac{y^2_{t-2}}{\sqrt{w+\alpha y^2_{t-1}}}\right]$

and $H^*_{L13}: u_t \approx (y_t - \phi_1 y_{t-1} - \gamma y_{t-2})/\left[\sqrt{w + \alpha y^2_{t-1}} + \frac{\gamma}{2}\frac{y^2_{t-2}}{\sqrt{w+\alpha y^2_{t-1}}}\right]$

$\approx \frac{y_t - \phi_1 y_{t-1}}{\sqrt{w+\alpha y^2_{t-1}}} - \gamma\left[\frac{y_{t-2}}{\sqrt{w+\alpha y^2_{t-1}}} + 0.5\frac{(y_t-\phi_1 y_{t-1})y^2_{t-2}}{(w+\alpha y^2_{t-1})^{3/2}}\right] \approx g^*(\tilde{Y}_t; \theta, \gamma)$

Under the sequence of local alternatives, we consider doubly indexed sequences $(y_{T,t})$, i.e. a sequence of processes indexed by T.

In this framework, what matters is the local impact on the autocovariances, i.e.:

$$\Gamma(h; \theta_T, \gamma_T, f_0) \approx \Gamma(h; \theta_0, 0, f_0) + \frac{\partial\Gamma(h; \theta_0, 0, f_0)}{\partial\theta'}(\theta_T - \theta_0) + \frac{\partial\Gamma(h; \theta_0, 0, f_0)}{\partial\gamma'}(\gamma_T - \gamma_0),$$

with $\Gamma(h; \theta_0, 0, f_0) = 0$. This leads to a local alternative written on the autocovariance:

$$\Gamma(h; \theta_T, \gamma_T, f_0) = \Delta(h; \theta_0, f_0, \mu, \nu)/\sqrt{T} \tag{3.9}$$

with

$$\Delta(h; \theta_0, f_0, \mu, \nu) = \frac{\partial\Gamma(h; \theta_0, 0, f_0)}{\partial\theta'}\mu + \frac{\partial\Gamma(h; \theta_0, 0, f_0)}{\partial\gamma'}\nu. \tag{3.10}$$

Its vec representation is denoted by $\delta(h;\theta_0, f_0, \mu, \nu) = vec\Delta(h;\theta_0, f_0, \mu, \nu)$.
Then, the asymptotic local power of the test, given f_0 fixed, is

$$\lim_{T\to\infty} P_{\theta_T,\gamma_T,f_0}[\hat{\xi}_T(H) > \chi^2_{1-\alpha}(K^2H - dim\theta)] = \beta(\theta_0, f_0, \mu, \nu; \alpha),$$

for any μ, ν, α and $(\theta_0, f_0) \in \Theta_0$.

Proposition 1: Under the sequence of local alternatives,
i) The autocovariance estimator

$$\hat{\Gamma}_T(h;\theta) = \frac{1}{T}\sum_{t=1}^{T} g(y_{T,t};\theta)g'(y_{T,t-h};\theta) - \frac{1}{T}\sum_{t=1}^{T} g(y_{T,t};\theta)'\frac{1}{T}\sum_{t=1}^{T} g(y_{T,t};\theta)$$

converges in probability to

$$\hat{\Gamma}_T(h;\theta) \to \Gamma(h;\theta_0, 0, f_0, \theta),$$

for all θ, h where the limit is computed under the null hypothesis. This convergence is uniform in θ.
ii) the GCov estimator converges in probability to θ_0:

$$\hat{\theta}_T \to \theta_0$$

Proof: The proof is based on the Law of Large Numbers (LLN) for doubly indexed sequences [see e.g. Andrews (1988), Newey (1991) and Appendix B for regularity conditions). The LLN implies the convergence of the estimated autocovariances.

Let us now discuss the convergence of the GCov estimator. The objective function $L_T(\theta)$

$$L_T(\theta) = \sum_{h=1}^{H} Tr[\hat{\Gamma}_T(h;\theta)\hat{\Gamma}_T(0;\theta)^{-1}\hat{\Gamma}_T(h;\theta)'\hat{\Gamma}_T(0;\theta)^{-1}] \tag{3.11}$$

tends to

$$L_\infty(\theta) = \sum_{h=1}^{H} Tr[\Gamma(h;\theta_0, 0, f_0, \theta)\Gamma(0;\theta_0, 0, f_0, \theta)^{-1}\Gamma(h;\theta_0, 0, f_0, \theta)'\Gamma(0;\theta_0, 0, f_0, \theta)^{-1}]. \tag{3.12}$$

This limit is the same as the limit of the objective function under the null hypothesis. Then, the consistency is proven as in Gourieroux, Jasiak (2022).

Let us now derive the distribution of the test statistic computed from the residuals of the model $g(y_t, \theta) = u_t$ estimated by the GCov estimator under the sequence of local alternatives.

The proof is given in Appendix B and based on the Central Limit Theorem (CLT) for doubly indexed sequences [see e.g. Wooldridge, White (1988) and Appendix B] .

Proposition 2: Let us consider the specification test of the null hypothesis:

$$H_0 \approx \Theta_0 = \{\theta, f : \Gamma(h;\theta, f) = 0 \ \ \forall h = 1, ..., H\},$$

against the sequence of local alternatives:

$$H_{1,T} = \Theta_{1,T} = \{\theta = \theta_0 + \mu/\sqrt{T}, \gamma = \nu/\sqrt{T}, f = f_0, \text{ with } (\theta_0, f_0) \in \Theta_0\}.$$

The expansion of the test statistic under the sequence of local alternatives is:

$$\hat{\xi}_T(H) = T\sum_{h=1}^{H}\{vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\} + o_p(1), \quad (3.13)$$

where

$$\begin{aligned}\Pi(h;\theta_0,f_0) \;\; &= \;\; [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}] - [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta'} \\ &\left\{\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta}\right\}^{-1} \\ &\times\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta'}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}].\end{aligned}$$

Then, under the sequence of local alternatives, $\hat{\xi}_T(H) \overset{a}{\sim} \chi^2(K^2H - dim\theta, \lambda(\theta_0, f_0, \mu, \nu))$, where the non-centrality parameter is

$$\lambda(\theta_0, f_0, \mu, \nu) = \sum_{h=1}^{H}\delta(h;\theta_0,f_0,\mu,\nu)'\Pi(h;\theta_0,f_0)\delta(h;\theta_0,f_0,\mu,\nu)),$$

with $\delta(h;\theta_0,f_0,\mu,\nu) = \frac{\partial\Gamma(h;\theta_0,0,f_0)}{\partial\theta'}\mu + \frac{\partial\Gamma(h,\theta_0,0,f_0)'}{\partial\gamma'}\nu.$
Proof: The proof or Proposition 2 is given in Appendix B.

Let the cumulative distribution function (c.d.f.) of the non-central chi-square distribution be denoted by $F(x;\kappa,\lambda)$, where κ denotes the degrees of freedom and λ is the non-centrality parameter. Moreover, $F(x;\kappa,\lambda) = 1-Q_{\kappa/2}(\sqrt{\lambda},\sqrt{x})$, where $Q_\delta(a,b)$ is a Marcum Q-function. For positive integer values of δ it is defined as:

$$Q_\delta(a,b) = \begin{cases} H_\delta(a,b) & a < b, \\ 0.5 + H_\delta(a,a) & a = b, \\ 1 + H_\delta(a,b) & a > b, \end{cases}$$

where $H_\delta(a,b) = \frac{\zeta^{1-\delta}}{2\pi} exp(-\frac{a^2+b^2}{2}) \int_0^{2\pi} \frac{cos(\delta-1)w - \zeta cos\,\delta\, w}{1-2\zeta cos w+\zeta^2} exp(ab\,cos\,w)dw$ and $\zeta = a/b$. Then, from Proposition 1 it follows that the local asymptotic power is given by:

$$\beta(\theta_0, f_0, \mu, \nu; \alpha) = Q_{(K^2H-dim\theta)/2}[\sqrt{\lambda(\theta_0, f_0, \mu, \nu)}, \sqrt{\chi^2_{1-\alpha}(K^2H - dim\theta)}].$$

From the monotonicity property of the Q-function, it follows that the asymptotic power function is a strictly decreasing function of the non-centrality parameter.

3.4 Bootstrap-Based Specification Test

The statistic $\hat{\xi}_T(H)$ is no longer asymptotically χ^2-distributed when the model parameters are estimated by an estimator different from the semi-parametric GCov estimator [see, Gourieroux, Jasiak (2022)]. If a different estimator $\tilde{\theta}_T$ of θ is used, such as the Quasi-Maximum Likelihood (QML), Approximate Maximum Likelihood (AML), or Maximum Likelihood (ML) estimator, the limiting distribution of the test statistic can be found by bootstrap. The test statistic becomes:

$$\tilde{\xi}_T(H) = \xi(H; \tilde{\theta}_T, \hat{f}_T). \tag{3.14}$$

where

$$\xi(H, \theta, f) = \sum_{h=1}^{H} Tr\left[\Gamma(h;\theta,f)\Gamma(0;\theta,f)^{-1}\Gamma(h;\theta,f)'\Gamma(0;\theta,f)^{-1}\right]$$

and it is computed from the sample autocovariances:

$$\hat{\Gamma}_T(h;\tilde{\theta}_T) = \frac{1}{T}\sum_{t=1}^{T} g(\tilde{Y}_t, \tilde{\theta}_T) g(\tilde{Y}_t, \tilde{\theta}_T)' - \frac{1}{T}\sum_{t=1}^{T} g(\tilde{Y}_t, \tilde{\theta}_T)' \frac{1}{T}\sum_{t=1}^{T} g(\tilde{Y}_t, \tilde{\theta}_T)$$

Then, the test can be applied by using the asymptotically valid critical value found by bootstrap. To approximate the distribution of $\tilde{\xi}_T(H)$ by bootstrap we could consider three different approaches by accommodating in different ways the bootstrapped estimators of the dynamic parameters and error density. The test statistic $\xi(H, \tilde{\theta}^s_T, \hat{f}^s_T)$ relies on $\tilde{\theta}^s_T$ and f^s_T both computed from the bootstrapped values $y^s_1, ..., y^s_T$. The test statistic $\xi(H, \tilde{\theta}_T, \hat{f}^s_T)$ relies on $\tilde{\theta}_T$ computed from the observations $y_1, ..., y_T$ and $\hat{f}^s_T$ computed from the bootstrapped values $y^s_1, ..., y^s_T$. The test statistic $\xi(H, \tilde{\theta}^s_T, \hat{f}_T)$ involves $\tilde{\theta}^s_T$ estimated from the bootstrapped values and $\hat{f}_T$ computed from the observations $y_1, ..., y_T$.

The most straightforward approach is to use the test statistic $\xi(H, \tilde{\theta}^s_T, \hat{f}^s_T)$ involving the bootstrapped values $y^s_1, ..., y^s_T$. This approach is followed in the empirical section later on.

Then, the critical value of the test statistic $\tilde{\xi}_T(H)$ can be found by bootstrap along the following steps:

1. Draw randomly $m(T)$ residuals $\hat{u}_t^s, t = 1, ..., m(T)$ from residuals $\hat{u}_t = g(\bar{Y}_t, \tilde{\theta}_T), t = 1, ..., T$ [5].

2. Build the bootstrapped time series of length $m(T)$: $y_t^s = c(\bar{Y}_{t-1}^s, \hat{u}_t^s, \tilde{\theta}_T), t = 1, ..., m(T)$.

3. Re-estimate the model parameter vector θ from $y_t^s,\ t = 1, ..., m(T)$, providing $\tilde{\theta}_T^s$.

Then, under the standard regularity conditions, it could be shown that the asymptotic distribution of $\sqrt{m(T)}(\hat{\theta}_T^s - \hat{\theta}_T)$ conditional on $\hat{\theta}_T$ is the same as the asymptotic distribution of $\sqrt{T}(\hat{\theta}_T - \theta_0)$. In particular, the bootstrap tests of hypotheses on θ are asymptotically valid.

4. Compute the test statistic $\tilde{\xi}_T^s(H)$ from $y_t^s, t = 1, ..., m(T)$,its nonlinear transforms and $\tilde{\theta}_T^s$.

5. Repeat the procedure S times.

6. Rank the test statistics $\tilde{\xi}_T^s(H),\ s = 1, ..., S$ and use the 95th percentile $\hat{q}_{95\%}$, say, as the critical value for testing the null hypothesis of absence of nonlinear or linear dependence.

Then, the null hypothesis is rejected if:

$$\hat{\xi}_T > \hat{q}_{95\%}$$

and not rejected, otherwise.

The validity of the bootstrap-based test can be established case by case. The performance of the bootstrap test can be illustrated for selected models and error distributions.

Alternatively, we can consider the regenerative block bootstrap (RBB) [Gourieroux, Jasiak (2019)]. The difference between the RBB and the previous approach is in step one, which is replaced by the following steps to build the $\hat{u}_t^s$ series.

i. Fix $\bar{u}$ equal to median of $\hat{u}_t$.

ii. Drive the following regenerative blocks:

$$\hat{\epsilon}_1 = (\hat{u}_{S_1}, ..., \hat{u}_{S_2-1}), \hat{\epsilon}_2 = (\hat{u}_{S_2}, ..., \hat{u}_{S_3-1}), ..., \hat{\epsilon}_{K_T-1} = (\hat{u}_{S_{K_T-1}}, ..., \hat{u}_{S_{K_t}-1}),$$

where S_1 to S_{K_T} are the crossing time of $\hat{u}_t$ over $\bar{u}$ and K_T is number of these crosses.

iii. Draw (with replacement) K_s blocks $\hat{\epsilon}_k^s, k = 1, ..., K$ from $(\hat{\epsilon}_1, ..., \hat{\epsilon}_{K_T-1})$, concatenate them until the length of $\hat{\epsilon}^s = (\hat{\epsilon}_1^s, \hat{\epsilon}_2^s, ..., \hat{\epsilon}_{K_s}^s)$ be more or equal to $m(T)$ and right censor it to get $m(T)$ observations $\hat{u}_t^s,\ t = 1, ..., m(T)$.

[5] Davis, Wu (1997) suggest using $T, m(T) \to \infty$ and $m(T)/T \to 0$.

The rest of the RBB approach follows the steps 2 to 6 given above.

4 Finite Sample Performance of Tests of Absence of (Non)Linear Dependence and Specification Tests

This section examines the finite sample performance of the proposed test statistics in selected causal-noncausal processes. We perform simulations to study their empirical size and power. The test stastistic (2.7) for testing for the absence of (non)linear dependence in time series is discussed in Section 4.2.2. Section 4.2.3 covers the performance of the GCov-based specification test statistic (3.7). The performance of the bootstrap test (3.9) is illustrated in Section 4.2.4.

As the specification tests in Sections 4.2.3 and 4.3.3 are applied to the residuals of estimated models, Section 4.1 reviews the estimation methods available for semi-parametric and fully parametric causal-noncausal models.

4.1 Estimation of Causal-Noncausal Processes

The estimation of univariate causal-noncausal univariate MAR(r,s) process defined in Section 3.1, Example 2 and multivariate causal-noncausal VAR model given in Example 3, Section 3.1 can be based either on a semi-parametric approach introduced by Gourieroux, Jasiak (2023), or the Maximum Likelihood method. These two methods are discussed below.

4.1.1 Semi-Parametric Estimation

Under the semi-parametric approach introduced by Gourieroux, Jasiak (2023), the noncausal model is a special case of model (3.1) with parameter vector θ. The model is considered semi-parametric, and the error density is left unspecified.

The GCov estimator of θ is given in equations (3.5)-(3.6). For a given transformation a, the GCov can be written as:

$$\hat{\theta}_T(H) = Argmin_\theta \sum_{h=1}^{H} Tr[\hat{R}_a^2(h,\theta)],$$

where

$$\hat{R}_a^2(h,\theta) = \hat{\Gamma}_a(h;\theta)\hat{\Gamma}_a(0,\theta)^{-1}\hat{\Gamma}_a(h;\theta)'\hat{\Gamma}_a(0;\theta)^{-1},$$

and $\hat{\Gamma}_a(h;\theta)$ is the sample covariance between $(a(u_t;\theta))$ and $a(u_{t-h};\theta)$.

The GCov estimator has no closed-form expression in general. As mentioned in Section 3, under the regularity conditions given in Gourieroux, Jasiak (2023), the GCov estimator is consistent, and asymptotically normally distributed. It is also semi-parametrically efficient and can achieve full parametric efficiency for specific well-chosen transformations a.

4.1.2 Parametric Estimation

Alternatively, when the model is fully parametric, and the errors are assumed to follow a parametric density, the Maximum Likelihood (ML) estimation can be used for the parameters of noncausal (mixed) processes.

The estimation of univariate MAR(r,s) processes can be based on the Approximate Maximum Likelihood (AML) estimator introduced by Lanne, Saikkonen (2010).

Then, the sample used in the approximate likelihood is reduced to $T-(r+s)$ observations. The approximate likelihood disregards the first r state variables that summarize the effect of shocks before time r and the last s state variables that summarize the effect of shocks after time $T-s$ and is therefore constructed from errors $u_{r+1}, ...u_{T-s-1}$ only. The first error to be included in the likelihood function is u_{r+1}. The last error in the sample to be included in the likelihood function is: u_{T-s-1}.
The Approximate Maximum Likelihood (AML) estimator is defined as:

$$(\hat{\Psi}, \hat{\Phi}, \hat{\theta}) = Argmax_{\Psi,\Phi,\theta} \sum_{t=r+1}^{T-s} \ln f[\Psi(L^{-1})\Phi(L)y_t; \gamma],$$

where $f[.;\gamma]$ denotes the probability density function of u_t.

In the case when the true density function is unknown, the AML estimator can be unreliable [Hecq, Lieb and Telg (2016)]. In such a case, there is an advantage in using the semi-parametric GCov estimator and leave the density unspecified.

The ML estimator for Vector Autoregressive Causal-Noncausal Process with a parametric error density is introduces in Davis, Song (2020).

4.2 Simulation Study

We consider the univariate MAR(r,s) and mixed VAR(1) processes with i.i.d. errors with different distributions.

We generate univariate and multivariate causal-noncausal processes of sample sizes of $T = 100, 200, 500$.

For univariate processes, we consider i.i.d. errors with a uniform distribution $\mathcal{U}_{[-1,1]}$, Laplace distribution with mean zero and variance one and t-student with 4 degrees of freedom(t(4)) distributions. The variance of errors with $t(4)$ distribution is finite and equal to 2.

For multivariate processes, we follow the examples given in the literature and consider serially and cross-sectionally i.i.d. errors with the component error processes following student t(4), t(5) and t(6) densities with variances of 2, 3/5 and 2/3, respectively.

4.2.1 Data Generating Process

a) Univariate Noncausal Process

In the univariate framework, we apply the simulation method proposed in Gourieroux, Jasiak (2016) to generate the causal-noncausal processes. For example, we use equation 3.2 where s is the order of noncausal polynomial, r is order of causal polynomial and $p = r + s$ is the combined order of the MAR process. For $r = 0$ and $s = 1$, we generate the MAR(0,1), i.e. the noncausal autoregressive process of order 1

$$y_t = \psi_1 y_{t+1} + u_t \ \ , |\psi| < 1. \tag{3.15}$$

By setting $r = 1$ and $s = 1$, we generate MAR(1,1) The values of coefficients $\phi(z)$ and $\psi(z)$ considered are chosen so that the strict stationarity conditions are satisfied. When $\psi = 0$, the MAR(1,1) is a purely causal autoregressive process and if $\phi = 0$, it is a purely noncausal process. If both autoregressive polynomials contain non-zero coefficients, then equation (3.2) describes a mixed causal-noncausal MAR(1,1) process. The mixed process contains both leads and lags of y_t, and admits a two-sided moving average representation [Gourieroux, Zakoian (2015)].

Figure 1 shows the examples the trajectories of MAR(0,1) and MAR(1,1) processes for the three error distributions considered.

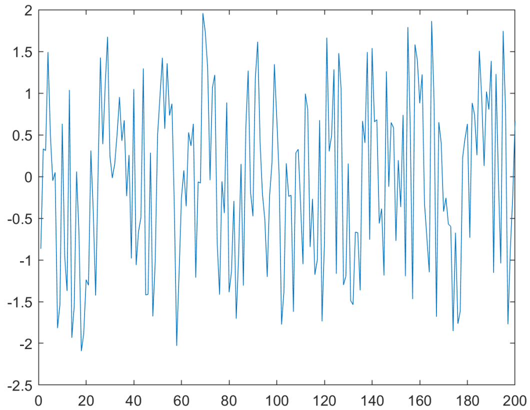
We observe that a large negative error value creates a spike in the trajectory with an explosion rate of about $1/\psi$ and a collapse rate of ϕ. In other words, we observe a jump if $\psi = 0$ and $\phi > 0$ and an explosive bubble if ψ is small, positive and $\phi = 0$.

b) Multivariate Noncausal Process

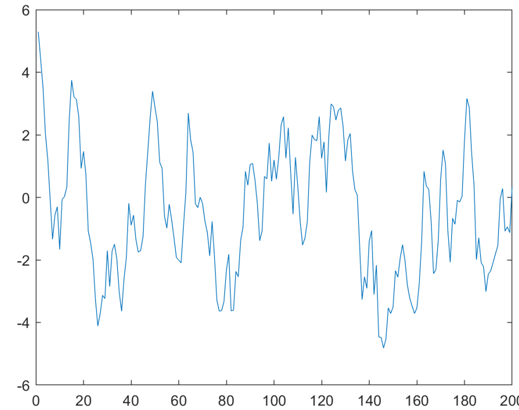
We generate three mixed bivariate VAR(1) models:

$$Y_t = \Phi Y_{t-1} + u_t, \tag{3.16}$$

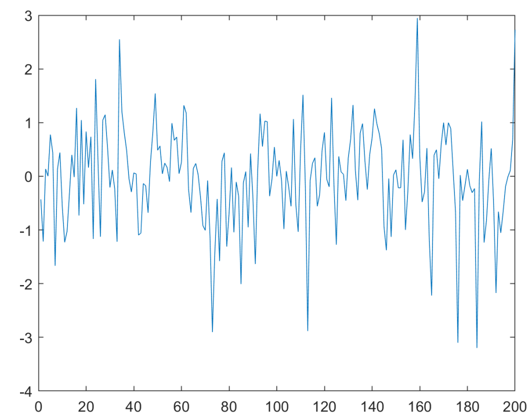
where the errors u_t are i.i.d. and have a t(4), t(5) or t(6) distributions, by using the simulation approach of Gourieroux, Jasiak (2016). The autoregressive parameters of these processes are set equal to those estimated in Gourieroux, Jasiak (2017), Davis, Song (2020), and Gourieroux, Jasiak (2022). Each of these VAR(1) models has one eigenvalue inside the unit-root circle and one outside. The 2×2 matrix Φ can be decomposed as:



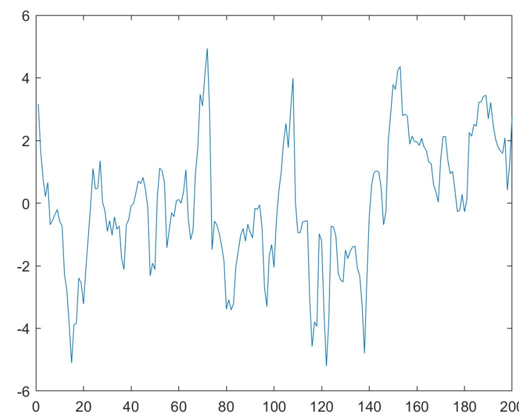
(a) noncausal AR(1), Uniform, $\psi_1 = 0.2$



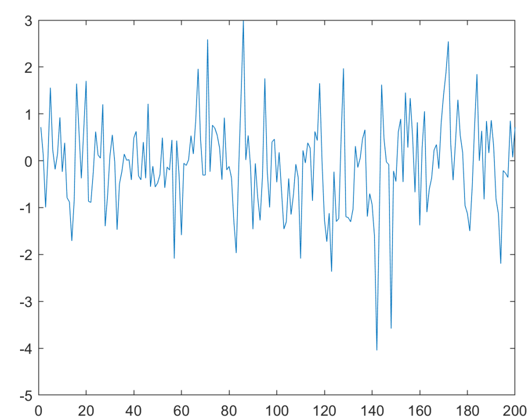
(b) MAR(1,1), U.,$\phi = 0.2, \psi = 0.8$



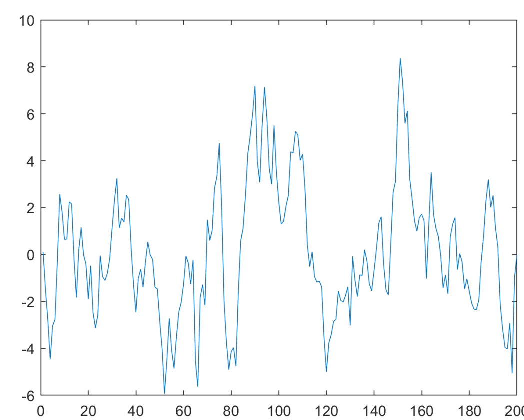
(c) noncausal AR(1), Laplace, $\psi_1 = 0.2$



(d) MAR(1,1), L.,$\phi = 0.2, \psi = 0.8$



(e) noncausal AR(1), t(4), $\psi_1 = 0.2$



(f) MAR(1,1), t(4) $\phi = 0.2, \psi = 0.8$

Figure 1: Plots of noncausal univariate processes, $T = 200$; L.:Laplace, U.:Uniform

$$\Phi = AJA^{-1} = A \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} A^{-1},$$

where the eigenvalues are denoted by λ_1 and λ_2, where $\lambda_1 < 1$ and $\lambda_2 > 1$. Matrix A and its inverse, A^{-1} contain the associated eigenvectors. We generate the VAR(1) processes with the following autoregressive parameter matrices:

Model S1:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0.7 & 0 \\ 0 & 2 \end{bmatrix}, \Phi = \begin{bmatrix} 0.7 & -1.3 \\ 0 & 2 \end{bmatrix},$$

Model S2:

$$A = \begin{bmatrix} 1 & 1 \\ 0.5 & 2 \end{bmatrix}, J = \begin{bmatrix} 0.5 & 0 \\ 0 & 2 \end{bmatrix}, \Phi = \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & 1.7 \end{bmatrix},$$

Model S3:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, J = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.2 \end{bmatrix}, \Phi = \begin{bmatrix} 0.9 & -0.3 \\ 0 & 1.2 \end{bmatrix},$$

4.2.2 Absence of (Non)linear Dependence Test for Time Series

This section examines the size and power of the test of the null hypothesis of absence of nonlinear dependence of y_t, based on the statistic (2.7) applied to a set of transformations a of a univariate time series $u_t = y_t$. We consider two transformations of the time series: y_t, y_t^2 and lag $H = 1$ where the true distribution f_0 is either Uniform, Laplace, or t(4).

It is easy to show that the test statistic considered is invariant with respect to the scale effect and change of sign of y_t^a. Let us consider a diagonal matrix A:

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^2 \end{bmatrix}$$

where λ represents the scale effect, or the change of sign effect for $\lambda = -1$. Then the multivariate R-square of process $y_t^a = [y_t, y_t^2]'$:

$$R_a^2(1) = \Gamma_a(1)\Gamma_a(0)^{-1}\Gamma_a(1)'\Gamma_a(0)^{-1}$$

computed for the rescaled process Ay_t^a is:

$$
\begin{aligned}
R_a^2(1) &= A\,\Gamma_a(1)\,A[A\,\Gamma_a(0)\,A]^{-1}\,A'\Gamma_a(1)'\,A'\,[A\,\Gamma_a(0)\,A]^{-1} \\
&= A\Gamma_a(1)AA^{-1}\Gamma_a(0)^{-1}A^{-1}A\Gamma_a(1)'AA^{-1}\Gamma_a(0)^{-1}A^{-1} \\
&= A\Gamma_a(1)\Gamma_a(0)^{-1}\Gamma_a(1)'\Gamma_a(0)^{-1}A^{-1} \qquad (3.17)
\end{aligned}
$$

Hence, we find that for multivariate R-square of the transformed process is $AR_a^2(1)A^{-1}$. We see that its trace is

$$Tr(AR_a^2(1)A^{-1}) = Tr(R_a^2(1)AA^{-1}) = TrR_a^2(1),$$

which implies that the test statistic $TTrR_a^2(1) = TTr(AR_a^2(1)A^{-1})$ remains unchanged and is equal for y_t^a and Ay_t .

The null hypothesis is the strong white noise hypothesis:

$$H_0 = (y_t = u_t)$$

The alternative hypothesis is either a MAR(1,0):

$$H_1 = ((1 - \gamma L)y_t = u_t),$$

or MAR(0,1) model

$$H_1 = ((1 - \gamma L^{-1})y_t = u_t),$$

respectively.

For each value of γ, where the coefficient γ varies between 0 and 1 by increments of 0.1, and each f_0, we simulate the series $y_1^s, ..., y_T^s, s = 1, ..S$. We use S=1000 replications. Then, we compute the test statistic from the simulated series and their nonlinear transformations for $T = 100, 200, 500$.

Let us now investigate the empirical size and power of the portmanteau test of absence of nonlinear dependence discussed in Section 2.2. The nominal size of the test is $\alpha = 0.05$.

We first consider a fixed alternative, assuming a specific error density as given. The first row of Table, 1 for the zero values of autoregressive coefficients provide the empirical size. The remaining rows of this Table illustrate the empirical power of the test with respect to fixed alternatives. The columns of this Table pertain to different sample sizes and the Uniform, Laplace and t(4) error distributions. The results for the size and power of the test

Table 1: MAR(0,1): Test of absence of (non)linear dependence. The first row ($\gamma = 0$) shows empirical size of test and the remaining rows show the power with respect to fixed alternatives.

γ	T=100			T=200			T=500		
	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)
0.0	0.037	0.053	0.050	0.042	0.055	0.050	0.048	0.053	0.052
0.1	0.073	0.073	0.075	0.145	0.135	0.128	0.375	0.352	0.344
0.2	0.233	0.218	0.235	0.541	0.540	0.534	0.957	0.956	0.964
0.3	0.564	0.564	0.568	0.923	0.929	0.935	0.999	1	1
0.4	0.860	0.873	0.887	0.997	0.997	0.998	1	1	1
0.5	0.976	0.984	0.986	0.999	1	1	1	1	1
0.6	0.998	0.999	0.999	1	1	1	1	1	1
0.7	0.999	1	1	1	1	1	1	1	1
0.8	1	1	1	1	1	1	1	1	1
0.9	1	1	1	1	1	1	1	1	1

Table 2: local results of MAR(0,1): Test of absence of (non)linear dependence. The first row ($\psi = 0$) shows empirical size of test and the remaining rows show the local power with respect to the local alternative $\psi = \frac{\delta}{\sqrt{T}}$.

ψ	T=100			T=200			T=500		
	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)
$\frac{0}{\sqrt{T}}$	0.037	0.053	0.050	0.042	0.055	0.050	0.048	0.053	0.052
$\frac{0.1}{\sqrt{T}}$	0.039	0.047	0.047	0.045	0.052	0.051	0.045	0.055	0.052
$\frac{0.2}{\sqrt{T}}$	0.038	0.053	0.045	0.043	0.056	0.049	0.050	0.053	0.053
$\frac{0.3}{\sqrt{T}}$	0.040	0.051	0.047	0.047	0.053	0.050	0.050	0.055	0.054
$\frac{0.4}{\sqrt{T}}$	0.042	0.049	0.051	0.047	0.054	0.053	0.054	0.060	0.059
$\frac{0.5}{\sqrt{T}}$	0.048	0.052	0.050	0.049	0.059	0.060	0.063	0.062	0.058
$\frac{0.6}{\sqrt{T}}$	0.049	0.056	0.048	0.054	0.060	0.059	0.066	0.068	0.062
$\frac{0.7}{\sqrt{T}}$	0.053	0.056	0.059	0.065	0.067	0.059	0.072	0.070	0.067
$\frac{0.8}{\sqrt{T}}$	0.059	0.059	0.064	0.068	0.074	0.065	0.077	0.079	0.072
$\frac{0.9}{\sqrt{T}}$	0.063	0.067	0.068	0.074	0.080	0.076	0.088	0.086	0.081

for the MAR(1,0) model under the alternative are given in Table 15 in Appendix C. They are consistent with the results given in this table.

The results of this simulation show good empirical size and power of the test with respect to fixed alternatives, given each density. We observe that higher values of the autoregressive coefficients increase the power of the test. When the sample size increases, the power converges to 1 and the size converges to 0.05.

Furthermore, we investigate the power of the test under the local alternatives by generating MAR(0,1) with coefficients equal to $\frac{\delta}{\sqrt{T}}$, where δ varies between 0 to 0.9. The results are provided in Table 2. Since we consider the local alternatives, we expect asymptotically the powers to be close to the significance level for small δ. However, for bigger δ we deviate further.

4.2.3 GCov-based Model Specification Test based on Residuals

This section examines the empirical size and power of the GCov specification test (3.7) for semi-parametric models discussed in Section 3. Under the null hypothesis, the model is either a MAR(0,1) with errors $u_t = (1-\psi L^{-1})y_t$, a MAR(1,1) with errors $u_t = (1-\phi L)(1-\psi L^{-1})y_t$, or a VAR(1) model with bivariate errors $u_t = g(y_t; \theta)$. The models are estimated by the GCov estimator with the lag length H=3 and K=2 i.e. we consider the residuals $\hat{u}_t$ and squared residuals $\hat{u}_t^2$, where $\hat{u}_t = (1 - \hat{\psi} L^{-1}) y_t$, or $\hat{u}_t = (1 - \hat{\phi} L)(1 - \hat{\psi} L^{-1}) y_t$, for the MAR(0,1) and MAR(1,1) processes under the null hypothesis ,respectively. The bivariate series of residuals is $\hat{u}_t = g(y_t; \hat{\theta})$ for the VAR(1) model.

Empirical size

The autocovariances of nonlinear transformations of the residuals of the estimated models are therefore $\Gamma(h; \theta, f)$. Then, the null hypotheses considered to study the size of the test is:

$$H_{0,a} = \{\Gamma_{0,a}(h; \psi, f) = 0, \ \forall h = 1, ..., H\},$$

for the MAR(0.1),

$$H_{0,a} = \{\Gamma_{0,a}(h; \phi, \psi, f) = 0, \ \forall h = 1, ..., H\},$$

for the MAR(1,1), and

$$H_{0,a} = \{\Gamma_{0,a}(h; \theta, f) = 0, \ \forall h = 1, ..., H\},$$

for the VAR(1) processes.

We generate the noncausal MAR(0,1), MAR(1,1) with coefficients varying between 0.0 and 0.1. All the results are based on S= 1000 replications. The nominal size is $\alpha = 0.05$.

We illustrate the empirical size of the specification test applied to the MAR(1,0) model under the null hypothesis with Uniform, Laplace and t(4) error distributions in Table 3. Table 4 shows the empirical size of specification test applied to the MAR(1,1) model. The columns of these Tables pertain to different sample sizes and the Uniform, Laplace and t(4) error distributions.

Table 3: MAR(0,1): Empirical size of specification test at 5% significance level

ψ	T=100			T=200			T=500		
	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)
0.1	0.032	0.051	0.049	0.039	0.059	0.061	0.047	0.062	0.065
0.2	0.027	0.045	0.044	0.036	0.057	0.056	0.042	0.060	0.065
0.3	0.024	0.045	0.045	0.035	0.056	0.056	0.040	0.058	0.064
0.4	0.020	0.042	0.045	0.032	0.054	0.061	0.043	0.058	0.064
0.5	0.020	0.039	0.044	0.038	0.052	0.053	0.047	0.062	0.062
0.6	0.021	0.041	0.043	0.035	0.057	0.055	0.043	0.061	0.064
0.7	0.020	0.040	0.039	0.031	0.053	0.049	0.049	0.061	0.057
0.8	0.018	0.041	0.040	0.027	0.054	0.052	0.042	0.059	0.064
0.9	0.021	0.040	0.038	0.029	0.052	0.048	0.037	0.058	0.061

According to these results, the GCov test is conservative at $T = 100$ in all cases. It remains conservative for the Uniform distribution for all sample sizes T. When the sample size increases, the empirical size of the GCov test approaches the nominal size for all distributions.

Table 4: MAR(1,1) Empirical size of specification test at 5% significance level

$\theta = (\phi, \psi)$	T=100			T=200			T=500		
	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)
(0.1,0.9)	0.005	0.021	0.022	0.006	0.027	0.024	0.029	0.054	0.049
(0.2,0.8)	0.007	0.013	0.017	0.009	0.038	0.039	0.029	0.060	0.066
(0.3,0.7)	0.004	0.015	0.032	0.016	0.036	0.041	0.042	0.047	0.049
(0.4,0.6)	0.008	0.018	0.021	0.015	0.029	0.049	0.024	0.049	0.046
(0.5,0.5)	0.006	0.023	0.027	0.017	0.034	0.026	0.035	0.039	0.061
(0.6,0.4)	0.004	0.024	0.025	0.007	0.034	0.042	0.020	0.055	0.049
(0.7,0.3)	0.005	0.023	0.020	0.009	0.035	0.034	0.020	0.060	0.064
(0.8,0.2)	0.007	0.025	0.025	0.007	0.041	0.031	0.022	0.051	0.049
(0.9,0.1)	0.005	0.026	0.024	0.008	0.037	0.036	0.026	0.055	0.041

The results from Table 4 are simuilar to those in Table 3. The test is conservative for the Uniform distribution for all sample sizes. For all error distributions, the empirical size improves with increasing T.

Next, we examine the size of the specification test statistic (3.7) in application to multivariate models. We consider the mixed bivariate VAR(1) models S1, S2 and S3 given in Section 4.2.1. with t(4) or t(6) error distributions, respectively. These models are estimated by the GCov estimator with the lag length H=3 and K=2 transformations. Next, the test statistic (3.7) is estimated from the residuals and replicated 1000 times. The nominal level is 0.05.

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Table **??** shows hat the GCov specification test for mixed VAR(1) model performs well in terms of error type I, and the empirical size approaches the nominal size when the sample size increases.

Power of the test

Next, we investigate the performance of the GCov specification test in terms of its power under the local alternatives.

In the univariate time series, an interesting way to investigate the empirical power of the GCov test is to consider the null hypothesis of MAR(0,1) and deviate from the null by adding a causal component and then by increasing that causal coefficient. This transforms the MAR(0,1) model under the null hypothesis into a MAR(1,1) model under the alternative.

Therefore the alternative hypothesis is:

$$H_{1a} = (1 - \phi L)(1 - \psi L^{-1})y_t = u_t$$

with $\theta = \psi, \gamma = \phi$.

The MAR(0,1) model is estimated by the GCov with the lag length is H=3 and the transformation $\hat{u}_t$ and $\hat{u}_t^2$ of number K=2. We use S=1000 replications and consider the nominal size of 0.05.

Let us first illustrate the power of the test under a fixed alternative.

In Table 5, we provide the results on the empirical power of the specification tests applied to a noncausal MAR(0,1) model with the coefficients equal to 0.3, or 0.7 given in column 1. Column 2 shows the additional causal coefficient $\gamma = \phi$ of the MAR(1,1) under the fixed alternative for each given error density, which increases from 0.1 to 0.9 by increments of 0.1. The remaining columns pertain to the different sample sizes and error distributions.

Table 5: MAR(0,1): Empirical power of specification test at 5% significance level

ψ	$\gamma=\phi$	T=100			T=200			T=500		
		Uniform	Laplace	t(4)	Uniform	Laplace	t(4)	Uniform	Laplace	t(4)
0.3	0.1	0.022	0.05	0.043	0.043	0.069	0.071	0.053	0.087	0.096
	0.2	0.033	0.064	0.072	0.093	0.094	0.095	0.195	0.176	0.213
	0.3	0.046	0.085	0.105	0.142	0.198	0.248	0.53	0.411	0.519
	0.4	0.073	0.163	0.212	0.234	0.34	0.424	0.84	0.779	0.85
	0.5	0.077	0.235	0.321	0.33	0.553	0.623	0.937	0.948	0.973
	0.6	0.102	0.307	0.395	0.33	0.595	0.667	0.963	0.98	0.983
	0.7	0.101	0.315	0.427	0.365	0.595	0.639	0.939	0.944	0.978
	0.8	0.121	0.287	0.356	0.384	0.532	0.625	0.933	0.963	0.964
	0.9	0.24	0.311	0.336	0.562	0.625	0.654	0.983	0.982	0.988
0.7	0.1	0.035	0.047	0.052	0.043	0.09	0.078	0.116	0.102	0.122
	0.2	0.038	0.065	0.098	0.128	0.158	0.191	0.400	0.451	0.456
	0.3	0.087	0.146	0.175	0.305	0.366	0.429	0.854	0.869	0.872
	0.4	0.174	0.328	0.38	0.574	0.698	0.757	0.992	0.992	0.993
	0.5	0.363	0.594	0.591	0.849	0.922	0.949	1	1	1
	0.6	0.617	0.821	0.825	0.97	0.99	0.996	1	1	1
	0.7	0.864	0.941	0.962	1	1	1	1	1	1
	0.8	0.98	0.992	0.991	1	1	1	1	1	1
	0.9	0.997	0.999	0.998	1	1	1	1	1	1

Table 5 indicates that the more we deviate from the null hypothesis, the higher the empirical power of the test.

Furthermore, we consider the MAR(0,1) process and study the power under the local alternatives. To do that we deviate from the null hypothesis of MAR(0,1) by adding parameter $\gamma = \phi$ and increasing ϕ gradually with $\phi = \frac{\delta}{\sqrt{T}}$.

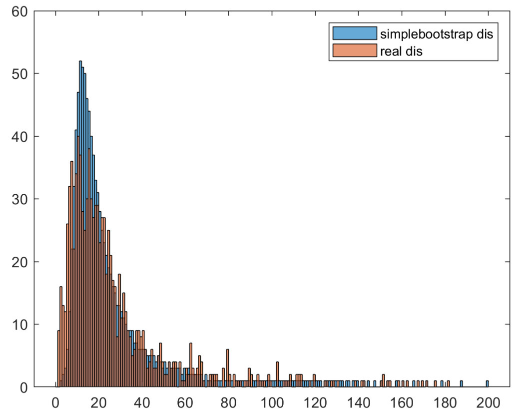
We use a similar approach to investigate the empirical power of the GCov test in the multivariate framework. We use the same models at error distributions as those investigated in the previous section on empirical size and illustrated in Table 5. To study the power under a fixed alternative the GCov test is computed for coefficients Φ^* that deviate from the true coefficient Φ. More specifically, we keep three coefficients $\phi_{12}, \phi_{21}, \phi_2$ of the Φ matrix fixed, and perturb ϕ_{11} by adding matrix P (2*2):

$$\Phi^* = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

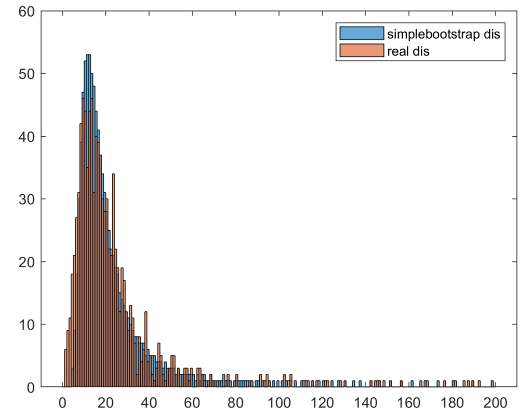
4.2.4 Bootstrap-Based Model Specification Test based on residuals

Let us now illustrate the finite sample performance of the bootstrap-based specification test statistic (3.9). Recall that the bootstrap test can be applied to test the specification of a model estimated by an estimator different than the GCov. We also compare the specification tests involving the simple bootstrap and block bootstrap given in Section 3.4.

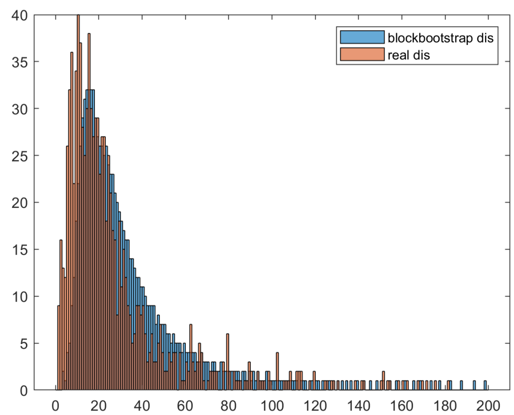
First, we consider a simulation experiment to compare the sample distributions of the bootstrap test statistic and the GCov test statistic in finite samples. We generate the data that follow a MAR(1,1) model with $t(3)$ error distribution and sample sizes T=100 and T=1000. The model is first estimated by the GCov with H=3 and K=2 transformations, which are the residuals and their squares and the GCov-based specification test statistic (3.7) is computed. Next, the model is estimated by an alternative estimator. We consider the Approximate Maximum Likelihood (AML) described in Section 4.1 with the t(3) -based log-likelihood function. Then, the bootstrap-based specification test statistic (3.9) is computed. The experiment is replicated 1000 times. We employ a simple bootstrap and block bootstrap methods with $S = 1000$ replications. Figure 2 presents the sample distribution of the GCov specification test statistic (3.7) for the simple and block bootstrap statistics (3.9) for two sample sizes.



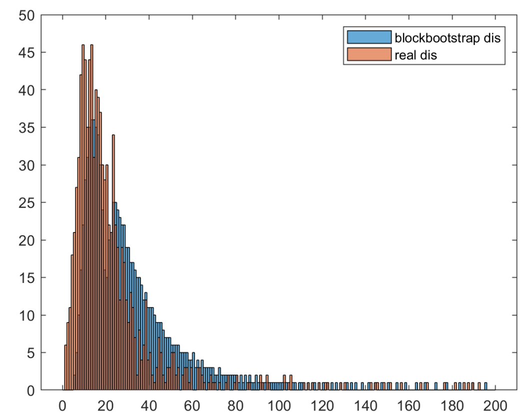
(a) simple bootstrap, $T = 100$



(b) simple bootstrap, $T = 1000$



(c) block bootstrap, $T = 100$



(d) block bootstrap, $T = 1000$

Figure 2: Plots of bootstrap-based test distribution, $\phi = 0.7$ and $\psi = 0.3$, 1000 replications

We find that the simple bootstrap sample distribution on average differs from the GCov-based specification test for close to 0 values of the test statistic. For the block bootstrap, the differences are mainly in the center of the distributions.

5 Empirical Application

In this section, we apply the GCov specification test to dynamic models fitted to the series of NFT token rates. We believe that the NFT token rates are both past and future dependent. We consider the daily rates of Decentraland (MANA), which has one of the highest market Cap among the NFTs. MANA is a cryptocurrency used to trade LAND, which is a NFT showing the ownership of digital real estate in a virtual reality. We use the sample of $T = 490$ daily rates of return of MANA from January 11, 2021, to May 15, 2022. We expect that this

Table 6: Test for absence of (non)linear dependence in data and MAR(1,1) residuals

tests	value of test	df	critical value
dependence test	48.46	12	21.02
GCov test	18.16	10	18.30

nonlinear pattern will be well accommodated by a causal-noncausal model.

To ensure the identification of the causal and noncausal roots of a causal-noncausal process can be distinguished, we test the data for normality. We apply the Kolmogorov-Smirnov normality test. The test statistic is 0.104 , which exceeds the critical value of 0.061. Hence the null hypothesis of normal distribution is rejected.

We explore the serial dependence of rates by using the test of absence of (non)linear dependence introduced in Section 2. We compute the test statistic (2.7) from the series using H=3 and K=2. The result is reported in the first row of Table 6 which indicates the existence of serial dependence in the data. This finding is confirmed by the the ACF of MANA rates and their squares.

We fit the causal-noncausal MAR models of a combined autoregressive order p=3 and varying causal and noncausal orders r and s. The estimated coefficients and the roots of the causal-noncausal polynomials are presented in Table 7 incolumns 1 to 3. Column 4 reports the value of the GCov test statistic of the specification test. We test the null hypothesis of strong white noise residuals in each model to assess its fit. Columns 5 and 6 provide the critical value of the chi-square distribution and the associated degrees of freedom. Under the strict stationarity assumption, all models have roots outside of the unit circle. We find that the MAR(2, 0) and MAR(3,0) models violate this assumption. According to the results of the GCov-based portmanteau test, only MAR(1,1) and MAR(1,2) do not reject the null hypothesis of strong white noise residuals, indicating the absence of (non)linear dependence in the residuals.

Among these two models, we select the MAR(1,1) because of the parsimony. Table 8 shows the values of the MAR(1,1) coefficients and their statistical significance.

The results of the GCov test are confirmed by the ACF of model residuals and their squares given in the two bottom panels of Figure 6, which are not statistically significant.

Table 7: Parameter Estimates for All the Casaul-Noncausal Models Up to $p = 3$.

models	parameters		roots	GCov test	df	$\chi^2(df)$
MAR(0,1)	ψ_1	0.12	**8.16**	37.96	11	19.67
MAR(1,0)	ϕ_1	-0.10	**-9.98**	40.74	11	19.67
MAR(0,2)	ψ_1	0.11	**1.27+4.55i**	36.96	10	18.30
	ψ_2	-0.04	**1.27-4.55i**			
MAR(1,1)	ψ_1	0.23	**4.32**	**18.16**	10	18.30
	ϕ_1	-0.20	**-4.90**			
MAR(2,0)	ϕ_1	4.13	**-4.94**	**18.16**	10	18.30
	ϕ_2	0.87	0.23			
MAR(1,2)	ψ_1	0.28	**1.76+3.05i**	**10.33**	9	16.91
	ψ_2	-0.08	**1.76-3.05i**			
	ϕ_1	-0.27	**-3.65**			
MAR(3,0)	ϕ_1	3.59	**-1.62+3.30i**	**14.85**	9	16.91
	ϕ_2	0.84	**-1.62-3.30i**			
	ϕ_3	0.28	0.26			

Table 8: Parameter Estimates, MAR(1,1)

parameters	estimate	st.err.	t-ratio
ψ	0.23	0.10	2.30
ϕ	-0.20	0.10	-1.96

6 Conclusion

This paper explored the finite sample performance of tests of the null hypothesis of absence of (non)linear dependence in time series and specification tests based on the residuals of a semi-parametric dynamic model. We described analytically the asymptotic distributions of the test statistics under the null hypothesis and local alternatives. Our simulation experiments show that the size and power of the test statistics are satisfactory. We applied the approach to the data on cryptocurrency returns and used the GCov-based test statistic to select the optimal fit.

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Appendix A

The following notations are used:

n - dimensions of Y_t

u_t is of dimension $J = dim(g)$

K - dimension of transformations a

$a(u_t) = g_a$ is of dimension K

Γ or (Γ^a) is of dimension K (except for Section 2.1.2, where Γ is of dimension n)

$dim(\theta)$ - dimension of θ

$dim(\gamma) = 1$, γ is a scalar

This Appendix considers an identity transformation $a = Id$. Hence, the dimensions n and K are equal: $n = K$.

Asymptotic Behavior of the Portmanteau Statistic Under the Independence Hypothesis

A.1 Asymptotic Behavior of Sample Autoregressive Coefficients

Suppose process Y_t follows a VAR(1) model:

$$Y_t = \alpha + BY_{t-1} + u_t, \tag{A.1}$$

where u_t is a square integrable strong white noise, $E(u_t) = 0$, $V(u_t) = \Sigma$ and Σ is invertible. This VAR model is a SUR model with identical regressors $X_t = Y_{t-1}$ in all equations. In this case, the OLS estimators applied equation by equation are equal to the GLS estimator of B [6]. Then, we have:

$$\hat{B} = \hat{\Gamma}(1)\hat{\Gamma}(0)^{-1} \iff \hat{B}' = \hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)'. \tag{A.2}$$

Moreover, we have asymptotically

$$\sqrt{T}[vec(\hat{B}') - vecB'] \approx N[0, \Sigma \otimes \Gamma(0)^{-1}], \tag{A.3}$$

where the $\otimes$ denotes the Kronecker product [see Chitturi (1974), eq. (1.13)]. In particular, under the null hypothesis $H_0 = (\Gamma(1) = 0) = (B = 0)$, we have $\Sigma = \Gamma(0)$ and

$$\sqrt{T}vec(B') \sim N(0, \Gamma(0) \otimes [\Gamma(0)^{-1}]). \tag{A.4}$$

[6] In this Appendix the index T of the estimators is omitted to simplify the notation.

A.2 Portmanteau Statistic as a Lagrange Multiplier test

The Lagrange Multiplier test statistic [7] for testing $H_0 = (\Gamma(1) = 0) = (B = 0)$ is:

$$\begin{aligned}
\hat{\xi}_T(1) &= Tvec[\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)']'[\hat{\Gamma}(0)^{-1} \otimes \hat{\Gamma}(0)]vec[\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)'] \\
&= Tvec[\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)']'[\hat{\Gamma}(0)^{-1/2} \otimes \hat{\Gamma}(0)^{1/2}][\hat{\Gamma}(0)^{-1/2} \otimes \hat{\Gamma}(0)^{1/2}]vec[\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)'] \\
&= Tvec[\hat{\Gamma}(0)^{-1/2}\hat{\Gamma}(1)'\hat{\Gamma}(0)^{-1/2}]'vec[\hat{\Gamma}(0)^{-1/2}\hat{\Gamma}(1)'\hat{\Gamma}(0)^{-1/2}],
\end{aligned}$$

by using the equality: $vec(ABC) = (C' \otimes A)vecB$ [see Lemma 4.3.1 in Horn, Johnson (1999) or Magnus, Neudecker (2019), ch. 18, p. 440-441]. Moreover, we have $[vecC]'[vecC] = Tr\ CC'$. Therefore,

$$\begin{aligned}
\hat{\xi}_T(1) &= T\ Tr[\hat{\Gamma}(0)^{-1/2}\hat{\Gamma}(1)'\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)\hat{\Gamma}(0)^{-1/2}] \\
&= T\ Tr[\hat{\Gamma}(1)'\hat{\Gamma}(0)^{-1}\hat{\Gamma}(1)\hat{\Gamma}(0)^{-1}] \\
&= T\ Tr\hat{R}^2(1). \qquad (A.5)
\end{aligned}$$

A.3 Asymptotic Behavior of Sample Autocovariance

Let us derive the asymptotic distribution of $\sqrt{T}vec[\hat{\Gamma}(1)' - \Gamma(1)']$ for model (A.1). We have:

$$\sqrt{T}vec[\hat{\Gamma}(1)' - \Gamma(1)'] = \hat{\Gamma}(0)\sqrt{T}[\hat{B}' - B'] \approx \Gamma(0)\sqrt{T}[\hat{B}' - B'] + o_p(1),$$

and then

$$vec[\sqrt{T}[\hat{\Gamma}(1)' - \Gamma(1)']] = vec[\Gamma(0)\sqrt{T}[\hat{B}' - B']] = [Id \otimes \Gamma(0)]vec(\sqrt{T}[\hat{B}' - B']) + o_p(1).$$

It follows that under the null hypothesis $H_0 : (B = 0)$ of independently and identically distributed (i.i.d.) process (Y_t) of dimension K with finite fourth order moments,:

$$\begin{aligned}
vec[\sqrt{T}[\hat{\Gamma}(1)' - \Gamma(1)']] &\approx N[0, [Id \otimes \Gamma(0)][\Gamma(0) \otimes \Gamma(0)^{-1}][Id \otimes \Gamma(0)]] \\
&= N[0, \Gamma(0) \otimes \Gamma(0)].
\end{aligned}$$

Under the null hypothesis the statistic (A.5) follows asymptotically a chi-square distribution $\chi^2(K^2)$, where $K = n$.

[7] This is a Lagrange Multiplier test statistic as the asymptotic covariance matrix of $vec(\hat{B}')$ is estimated under the null hypothesis [see Hosking (1981)].

A.4 Statistic Based on Several Autocovariances

The interpretation as a SUR regression can be extended to any lag H. Then, the VAR model becomes:

$$Y_t = \alpha + B_1 Y_{t-1} + \cdots + B_H Y_{t-H} + u_t, \tag{A.6}$$

where (u_t) is a square integrable strong white noise. Under the null hypothesis of independence of Y_t, or equivalently under $H_0 = \{B_1 = \cdots = B_H = 0\}$, the explanatory variables are orthogonal, and the OLS estimators of $B_1, ..., B_H$ are such that $\hat{B}_h$ coincides with the OLS estimator in the simple SUR model $Y_t = \alpha_h + B_h Y_{t-h} + v_t$. It follows that, under the null hypothesis, the estimators $\sqrt{T}\hat{B}_h,\ h = 1, ..., H$ are independent, normally distributed with the same distribution $N(0, \Gamma(0) \otimes \Gamma(0))$.
Then, the test statistics:

$$\hat{\xi}_T(H) \approx T \sum_{h=1}^{H} vec[\sqrt{T}\hat{\Gamma}(h)']'[\hat{\Gamma}_0(0)^{-1} \otimes \hat{\Gamma}_0(0)^{-1}] vec[\sqrt{T}\hat{\Gamma}(h)'] \tag{A.7}$$

follows asymptotically the chi-square distribution $\chi^2(K^2H)$, where $K = n$.

Appendix B
Asymptotic Distribution in the Semi-Parametric Framework

B.1 The Law of Large Numbers (LLN) for Triangular Arrays

As pointed out in Section 3.3.2, the proof of the consistency of estimated autocovariances and of the GCov estimator under local alternatives is similar to the proof under the null hypothesis of independence. The only difference is in the use of LLN for empirical autocovariances for a triangular array of observations, uniform in θ.

We give below a sufficient set of regularity conditions.

Regularity Conditions for LLN uniform in θ.

1. Conditions on the true nonlinear dynamics

i) The observations satisfy the model:

$$g^*(\tilde{Y}_{T,t}; \theta_T, \gamma_T) = u_t, \tag{B.1}$$

where the u_t's are i.i.d. with pdf f_0.

ii) The function g^* is invertible with respect to $Y_{T,t}$; then we can write:

$$Y_{T,t} = h(u_t, Y_{T,t-1}, ..., Y_{T,t-p}; \theta_T, \gamma_T). \quad \text{(B.2)}$$

iii) For each given T, $(Y_{T,t})$ with varying t, is a strictly stationary and ergodic solution of the autoregressive equation (B.2).

2. Conditions on the parameters

Suppose that the parameter space is $\Theta \times C$, where $\theta \in \Theta \subset \mathbf{R}^{dim(\theta)}$ and $\gamma \in C \subset \mathbf{R}$. We assume that:

i) Θ and C are compact sets with non-empty interiors.

ii) θ_0 is in the interior of Θ and 0 is in the interior of C.

iii) $\theta_T = \theta_0 + \mu/\sqrt{T},\ \gamma_T = \nu/\sqrt{T}$.

In particular, for T sufficiently large, θ_T, γ_T are in the interior of Θ and C, respectively.

3. Regularity conditions on function g^*

i) The functions $g_k^*(y; \theta, \gamma),\ k = 1, ..., K$ are continuously differentiable on the interior $\Theta \times C$.

ii) Let us define: $G_k^*(\tilde{y}) = Max_{(\theta,\gamma)\in\Theta\times C}[g_k^*(\tilde{y}, \theta, \gamma)]^2$. We assume $E_0 G_k^*(\tilde{Y}) < \infty,\ k = 1, ..., K$ where E_0 denotes the expectation computed for the process $(\tilde{Y}_t)$ associated with the "asymptotic" parameter values $(\theta_0, 0)$.

iii) Let us denote by $\mathcal{B}(\tilde{y})$ a uniform Lipschitz coefficient for functions $g_j^*(\tilde{y}; \theta, \gamma),\ j = 1, ..., J,\ g_j^{*2}(\tilde{y}; \theta, \gamma),\ j = 1, ..., J$ and for $g_j^*(\tilde{y}; \theta, \gamma),\ g_j^*(\tilde{y}_{-h}; \theta, \gamma),\ j, k = 1, ..., J, h = 1, ..., H$. In this expression $\tilde{y}$ denotes the trajectory of the process and $\tilde{y}_{-h}$ denotes this trajectory lagged by h. It is assumed that $sup_T \frac{1}{T} E|\mathcal{B}(Y_{T,t})] < \infty$ [Gourieroux, Jasiak (2022)], where the expectation is with respect to the distribution of process $Y_{T.} = (Y_{Tt})$, with varying t.

4. Condition of Near Epoch Dependence [De Jong (1988)]

The functions of the triangular array of random variables $Y_{T,t}, t \leq T, T \geq 1$ are $L_2 - NED$ (near epoch dependent), i.e. for $\nu(m) \geq 0$ and $c_{Tt} \geq 0$ and for all $m \geq 0$ and $t \geq 1$

$\sup_{\theta\in\Theta} E[g_j(\tilde{Y}_{T,t}, \theta) - E(g_j(\tilde{Y}_{T,t}, \theta)|Y_{T,t-m}, ..., Y_{T,t+m})]^2 \leq c_{T,t}\varphi(m)$

$\sup_{\theta\in\Theta} E[g_j^2(\tilde{Y}_{T,t}, \theta) - E(g_j^2(\tilde{Y}_{T,t}, \theta)|Y_{T,t-m}, ..., Y_{T,t+m})]^2 \leq c_{T,t}\varphi(m)$

$\sup_{\theta\in\Theta} E[g_j(\tilde{Y}_{T,t}, \theta) g_k(\tilde{Y}_{T,t-h}, \theta) - E(g_j(\tilde{Y}_{T,t}, \theta)|Y_{T,t-m}, ..., Y_{T,t+m}) E(g_k(\tilde{Y}_{T,t-h}, \theta)|Y_{T,t-m}, ..., Y_{T,t+m})]^2$
$\leq c_{T,t}\varphi(m)$

for all $j, k = 1, ..., K,\ h = 1, ..., H,\ \varphi(m) \to 0$ as $m \to \infty$ and $\limsup_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} c_{T,t} < \infty$.

From Assumption 4) it follows that functions $g(Y_{Tt})$ are uniformly integrable mixingales [DeJong (1998)]. Because the NED condition implies a mixingale condition, the weak LLN of Theorem 2, Andrews (1988) can be applied. Then theorem 4 of Andrews (1992) implies the uniform weak LLN (U-WLLN).

To summarize, we get the following Proposition:

Proposition B1:

Under the regularity conditions 1 to 4, we have:

$$plim_{T\to\infty}\hat{\Gamma}_T(h;\theta) = \Gamma_0(h;\theta)$$

uniformly in $\theta \in \Theta$ for $h = 1, ..., H$, where $\Gamma_0(h;\theta)$ is evaluated at θ_0, γ_0 and f_0 is the true pdf of the error.

When the functions g are distinguished from their transforms g_a, then conditions 1 and 2 concern g and conditions 3 and 4 concern g_a.

B.2 Central Limit Theorem (CLT) for Triangular Array

We need to introduce additional regularity conditions to justify the expansion (3.13) and the asymptotic normality of the sample autocovariances $\sqrt{T}\hat{\Gamma}(h;\theta_T,\gamma_T,f_0)$ computed under a sequence of local alternatives. To obtain the corresponding CLT, we use the conditional Lindeberg-Feller conditions for martingale difference triangular array [Dvoretski (1970), Brown (1971)] extended to the multivariate case [Kundu et al. (2000), Th. 1.3]. To apply these conditions, we need first to define the triangular filtration and the appropriate martingales. We denote by $\mathcal{F}_{T,t}$ the information generated by the array $Y_{T,\tau},\ \tau \leq t$. Then, we consider the different transformations $g_j^*(\tilde{Y}_{T,t};\theta_0,0),\ g_j^*(\tilde{Y}_{T,t};\theta_0,0)g_k^*(\tilde{Y}_{T,t-h};\theta_0,0),\ j,k = 1,...,K,\ h = 1,...,H$. They can be written as a vector $G^*(\tilde{Y}_{T,t};\theta_0,0)$, say. Next, we transform this vector into a multivariate martingale difference array by considering:

$$X_{T,t} = \frac{1}{\sqrt{T}}\{G(\tilde{Y}_{T,t};\theta_0,0) - E_0[G(\tilde{Y}_{T,t};\theta_0,0)|\mathcal{F}_{T,t-1}]\}.$$

The additional regularity conditions are the following:

Regularity Conditions for the CLT

i) The multivariate martingale difference array $X_{T,t}$ has finite second-order moments.

ii) For any vector b of the same dimension as $X_{T,t}$, there exists a matrix Ω such that:

$$\sum_{t=1}^{T} E[(b'X_{T,t})^2|\mathcal{F}_{T,t-1}] \xrightarrow{P} b'\Omega b.$$

iii) Conditional Lindeberg-Feller condition:

$$\sum_{t=1}^{T} E\{(b'X_{T,t})^2 \mathbf{1}_{|b'X_{T,t}|>\epsilon}|\mathcal{F}_{T,t-1}\} \xrightarrow{P} 0, \text{ for any } b \text{ and } \epsilon > 0.$$

These regularity conditions ensure that the sum $S_T = \sum_{t=1}^{T} X_{T,t}$ tends in distribution to the multivariate Gaussian distribution $N(0, \Omega)$. Then we get the asymptotic normality of the estimated autocovariances under the sequence of local alternatives by applying the Slutsky Theorem.

Proposition B2:

Under the sequence of local alternatives and the regularity conditions 1-5, the vectors $vec[\sqrt{T}\hat{\Gamma}_T(h; \theta_T, \gamma_T, f_0)]$ are asymptotically independent, normally distributed with mean $\Delta(h; \theta_0, f_0, \mu, \nu)$ defined in 3.10 and variance-covariance matrix $\Gamma_0(0, \theta_0) \otimes \Gamma_0(0, \theta_0)$

Thus the behavior of the estimated autocovariances differs from its behavior under the null by the presence of the asymptotic bias measured by $\Delta(h; \theta_0, f_0, \mu, \nu)$.

We have introduced a set of regularity conditions to derive the asymptotic behavior of the estimated autocovariances. Let us now explain why this set of conditions is also sufficient to derive the asymptotic behavior of the GCov estimator and of the portmanteau statistic. First, we review the standard expansions under the null hypothesis. Next, we derive their analogues under the sequence of alternatives, before applying the CLT to estimated autocovariances of a triangular array.

B.3 First-order Expansion of the GCov Estimator under the Null Hypothesis

Below we recall the results under the null hypothesis derived in Gourieroux, Jasiak (2022). Let us consider $H = 1$ for ease of exposition. The first-order conditions of the GCov estimator are

$$\frac{\partial Tr\hat{R}^2(1; \theta_j)}{\partial \theta_j} = 0, \quad j = 1, ..., J = dim\theta,$$

Let us define:

$$A(\theta_0) = 2\frac{\partial vec\Gamma(1; \theta_0)'}{\partial \theta}[\Gamma(0; \theta_0)^{-1} \otimes \Gamma(0; \theta_0)^{-1}],$$

and

$$J(\theta_0) = -2\frac{\partial vec\Gamma(1; \theta_0)'}{\partial \theta}[\Gamma(0; \theta_0)^{-1} \otimes \Gamma(0; \theta_0)^{-1}]\frac{\partial vec\Gamma(1; \theta_0)}{\partial \theta}.$$

The first-order Taylor series expansion of the GCov estimator is:

$$\sqrt{T}(\hat{\theta_T} - \theta_0) = J(\theta_0)^{-1}A(\theta_0)vec[\sqrt{T}\hat{\Gamma}_T(1; \theta_0)'] + o_p(1), \quad \text{(B.3)}$$

B.4 Expansion of the Portmanteau Statistic under the Null Hypothesis

The expansion of the test statistic under the null hypothesis is:

$$\hat{\xi}_T(H) = \sum_{h=1}^{H} vec[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)']'\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)'] + o_p(1), \qquad \text{(B.4)}$$

[See Gourieroux, Jasiak, "Generalized Covariance Estimator Supplemental Material" (2023), equation (a.11)], where

$$\begin{aligned}\Pi(h;\theta_0,f_0) &= [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}] - [\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta'}\\ &\left\{\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\frac{\partial vec\Gamma(h,\theta_0,f_0)}{\partial\theta}\right\}^{-1}\\ &\times\frac{\partial vec\Gamma(h,\theta_0,f_0)'}{\partial\theta'}[\Gamma_0(0,\theta_0,f_0)^{-1}\otimes\Gamma_0(0,\theta_0,f_0)^{-1}]\end{aligned}$$

Matrix $\Pi(h;\theta_0,f_0)$ satisfies for all $h=1,...,H$ the condition

$$\Pi(h;\theta_0,f_0)V_{asy}[\sqrt{T}\hat{\Gamma}_T(h,\theta_0)']\Pi(h;\theta_0,f_0) = \Pi(h;\theta_0,f_0)$$

where $V_{asy}[\sqrt{T}\hat{\Gamma}_T(h,\theta_0,f_0)'] = [\Gamma_0(0,\theta_0,f_0)\otimes\Gamma_0(0,\theta_0,f_0)]$.

This condition means that the matrix $\Pi(h;\theta_0,f_0)$ has an interpretation in terms of orthogonal projector. Therefore, under the null hypothesis, the quadratic form (A.10) where the $vec(\sqrt{T}\hat{\Gamma}_T(h;\theta_0,f_0))$ are independent identically distributed still follows a chi-square distribution with a reduced degree of freedom.

B.5 Asymptotic Behavior Under the Local Alternatives

Under the regularity conditions 1-6, it is easy to see that expansions similar to (B.3)-(B.4) are still valid under the sequence of local alternatives, by using the LLN for triangular arrays and the convergence of order $1/\sqrt{T}$ of the estimated autocovariances that follows from the CLT. For example, we have still the expansion:

$$\hat{\xi}_T(H) = T\sum_{h=1}^{H}\{vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\Pi(h;\theta_0,f_0)vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]\} + o_p(1)$$

similar to expansion (B.4) where now the vectors $vec[\sqrt{T}\hat{\Gamma}_T(h;\theta_T,\gamma_T,f_0)]$, $h=1,...,H$ are asymptotically independent with distribution

$N[\delta(h;\theta_0,f_0,\mu,\nu),\Gamma(0;\theta_0,f_0)\otimes\Gamma(0;\theta_0,f_0)]$.

by the CLT.

Then, under the sequence of local alternatives, the asymptotic distribution of $\hat{\xi}_T(H)$ is a chi-square distribution with non-centrality parameter λ:

$$\lambda(\theta_0, f_0, \mu, \nu) = \sum_{h=1}^{H} \delta(h, \theta_0, f_0, \mu, \nu)' \Pi(h; \theta_0, f_0) \delta(h, \theta_0, f_0, \mu, \nu)$$

and a degree of freedom equal to the rank of matrix $\Pi(H; \theta_0, f_0) = diag[\Pi(h; \theta_0, f_0)]$, where diag denotes a diagonal matrix.

B.6 The Behavior of Test of Independence under Local Alternatives

The results of Section B.5 can be applied to a null hypothesis $H_0 = (y_t = u_t)$ without parameter θ and other forms of local alternatives.

Let us consider the test of absence of linear dependence in time series $y_t = u_t, t = 1, ..., T$ against local alternatives of autoregressive form. More specifically, we test

$$H_0 : \{\Gamma_0(h) = 0, \forall h = 1, ..., H\} = \{B_1 = \cdots = B_H = 0\},$$

against the local alternatives. The local alternatives can be defined in terms of autoregressive parameters $B_1, ..., B_H$, or equivalently in terms of autocovariances $\Gamma(h), h = 1, ..., H$. Thus, the additional parameter γ is not necessarily a scalar. We follow the latter approach with the sequence of local alternatives:

$$H_{1,T} = \{\Gamma_T(h) = \Delta(h)/\sqrt{T},\ h = 1, ..., H\} = \{vec\Gamma_T(h) = \delta(h)/\sqrt{T},\ h = 1, ..., H\},$$

with $\delta(h) = vec\Delta(h)$.

Under the sequence of local alternatives, the estimated autocovariances are asymptotically independent with asymptotic normal distributions.

$$vec[\sqrt{T}\hat{\Gamma}_T(h)'] \overset{a}{\sim} N[\delta(h), \Gamma(0) \otimes \Gamma(0)].$$

Hence:

$$[\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}] vec[\sqrt{T}\hat{\Gamma}_T(h)'] \overset{a}{\sim} N[(\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2})\delta(h), Id]$$

It follows from the portmanteau statistic $\hat{\xi}_T(H)$ follows asymptotically under the sequence of local alternatives a chi-square $\chi^2(K^2 H, \lambda)$ distribution with non-centrality parameter λ, where

$$\lambda = \sum_{h=1}^{H} \delta(h)'[\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}][\Gamma(0)^{-1/2} \otimes \Gamma(0)^{-1/2}]\delta(h) = \sum_{h=1}^{H} \delta(h)'[\Gamma(0)^{-1} \otimes \Gamma(0)^{-1}]\delta(h), \tag{B.5}$$

is the non-centrality parameter.

Appendix C

Table 9: MAR(1,1) Empirical size of specification test at 5% significance level

$\theta = (\phi, \psi)$	T=100			T=200			T=500		
	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)
(0.1, 0.9)	0.112	0.054	0.016	0.117	0.057	0.030	0.108	0.069	0.047
(0.2, 0.8)	0.098	0.053	0.022	0.086	0.055	0.026	0.100	0.074	0.057
(0.3, 0.7)	0.093	0.035	0.015	0.098	0.055	0.028	0.062	0.066	0.056
(0.4, 0.6)	0.095	0.057	0.020	0.084	0.060	0.027	0.072	0.075	0.038
(0.5, 0.5)	0.084	0.043	0.024	0.091	0.059	0.029	0.072	0.067	0.039
(0.6, 0.4)	0.083	0.054	0.021	0.069	0.061	0.027	0.071	0.062	0.047
(0.7, 0.3)	0.080	0.038	0.026	0.085	0.050	0.036	0.071	0.075	0.059
(0.8, 0.2)	0.088	0.051	0.017	0.091	0.061	0.025	0.075	0.074	0.047
(0.9, 0.1)	0.098	0.046	0.016	0.128	0.061	0.036	0.132	0.068	0.050

Table 10: MAR(0,1): Test of absence of (non)linear dependence. The first row ($\gamma = 0$) shows empirical size of test and the remaining rows show the power with respect to fixed alternatives.

γ	T=100			T=200			T=500		
	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)
0	0.0625	0.0569	0.0480	0.0588	0.0583	0.0524	0.0444	0.0528	0.0550
0.1	0.2920	0.0982	0.0665	0.4152	0.1699	0.1267	0.6657	0.4041	0.3493
0.2	0.5617	0.3109	0.2245	0.8100	0.6370	0.5406	0.9982	0.9828	0.9623
0.3	0.8418	0.6850	0.5602	0.9935	0.9640	0.9332	1	1	0.9999
0.4	0.9798	0.9386	0.8842	0.9998	0.9996	0.9979	1	1	1
0.5	0.9991	0.9937	0.9860	0.9999	1	1	1	1	1
0.6	1	0.9998	0.9990	1	1	1	1	1	1
0.7	1	1	1	1	1	1	1	1	1
0.8	1	1	1	1	1	1	1	1	1
0.9	1	1	1	1	1	1	1	1	1

Table 11: local results of MAR(1,0): Test of absence of (non)linear dependence. The first row ($\phi = 0$) shows empirical size of test and the remaining rows show the local powerto fixed δ and $\phi = \frac{\delta}{\sqrt{T}}$.

δ	T=100			T=200			T=500		
	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)	t(1)	t(2)	t(5)
0	0.105	0.074	0.06	0.093	0.072	0.045	0.073	0.079	0.056
0.1	0.105	0.073	0.057	0.103	0.077	0.067	0.097	0.077	0.063
0.2	0.122	0.067	0.057	0.124	0.073	0.056	0.104	0.074	0.075
0.3	0.152	0.078	0.051	0.137	0.073	0.06	0.132	0.07	0.088
0.4	0.149	0.066	0.044	0.142	0.082	0.058	0.153	0.072	0.073
0.5	0.17	0.073	0.053	0.187	0.07	0.059	0.145	0.091	0.057
0.6	0.149	0.079	0.046	0.171	0.083	0.057	0.21	0.076	0.07
0.7	0.173	0.066	0.046	0.205	0.101	0.074	0.171	0.083	0.058
0.8	0.199	0.088	0.052	0.177	0.091	0.065	0.204	0.091	0.064
0.9	0.231	0.079	0.054	0.248	0.095	0.066	0.206	0.095	0.085