

Bubble Detection with Application to Green Bubbles: A Noncausal Approach

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Abstract

This paper introduces a new approach to detect bubbles based on mixed causal and noncausal autoregressive processes and their tail process representation during explosive episodes. Departing from traditional definitions of bubbles as nonstationary and temporarily explosive processes, we adopt a perspective in which prices are assumed to follow a strictly stationary process, with the bubble considered an intrinsic component of its nonlinear dynamics. We illustrate our approach with the phenomenon called the "green bubble" in the field of renewable energy investment.

Keywords: Energy finance, GCov estimator, Green bubbles, Non-Gaussianity, Noncausal models, Renewable energy investment

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1 Introduction

This paper introduces a novel method of bubble detection based on strictly stationary non-causal autoregressive processes and their tail process representation during a bubble. The tail process was first defined by [Basrak and Segers \(2009\)](#) as the weak limit of finite-dimensional distributions of the time series, given that an extreme value occurs. This definition was further generalized to the tail empirical process by [Kulik and Soulier \(2020\)](#), which is based solely on observations exceeding a high threshold, and interpreted by the authors as "a model for the clusters of exceedances". It is valid for geometrically ergodic Markov processes, and consequently also for mixed causal-noncausal autoregressive $MAR(1,1)$ and purely noncausal $MAR(0,1)$ processes – where the notation $MAR(r, s)$ indicates a mixed model where r is the causal order and s the noncausal one – with locally explosive patterns including bubbles and spikes. As shown, for example, by [Fries and Zakoian \(2019\)](#) and [Cavaliere et al. \(2020\)](#), the $MAR(1,1)$ and $MAR(0,1)$ processes are well-suited for modeling variables such as commodity and cryptocurrency prices. In this context, the paper introduces a simple statistic with an asymptotic normal distribution that provides an ex ante warning of a potentially emerging bubble. Ex post, this statistic is a tool to measure the duration of the bubble and identify its starting and ending dates.

MAR processes have recently received attention in applications to bubble forecasting and testing. [de Truchis et al. \(2025\)](#) considers the forecasting of extreme trajectories of α -stable MAR processes, based on a measure describing the conditional distribution of a normalized path of the process, following a large value. The method is applied to predict the occurrences of the El Niño and La Niña phases of the temperatures and winds of the Pacific Ocean. In comparison, our approach focuses on testing rather than predicting the bubbles and is computationally simple as it does not require determining the length of a future path as a tuning parameter or normalization. [Blasques et al. \(2025\)](#) introduces a bubble test based on the detection of a large future shock in the forward-looking component of an MAR process and applies it to oil prices. Compared to their approach, our proposed test statistic has the advantage of being easy to compute and having a known normal asymptotic distribution, rather than depending on the error distribution of a given model.

Alternative methods for detecting bubbles, such as the Dickey-Fuller augmented test (e.g., SADF and GSADF), assess unit roots and explosive regimes [[Phillips et al. \(2011\)](#); [Phillips et al. \(2015\)](#)]. However, our approach departs from these conventional methods by employing mixed autoregressive causal-noncausal models, which capture locally explosive patterns observed in the data. Unlike those detection approaches, we assume that the series

follows a strictly stationary process in which bubbles are an inherent part of the dynamics, rather than distinguishing a stationary and non-stationary (unit root) regime of a time series. In addition, it accommodates local explosive patterns with various rates of explosion, rather than a fixed rate of growth of a unit root process.

The literature on green energy stocks is relatively recent and focuses on return analysis. The pioneering paper by [Henriques and Sadorsky \(2008\)](#) shows that returns on technology stocks and oil prices are both Granger-caused by the returns of green (alternative) energy stocks, based on a vector autoregressive (VAR) model. The relationship between green stocks and oil has also been examined by [Sadorsky \(2012a\)](#), [Sadorsky \(2012b\)](#), and [Kumar et al. \(2012\)](#). These articles consider carbon prices and do not find their significant impact on green energy stocks. In contrast, this impact is evidenced after the year 2007 by [Managi and Okimoto \(2013\)](#). The literature has not yet considered the green stock price dynamics or detected and explained the presence of bubbles, which is done in this paper.

The rest of the paper is organized as follows. Section 2 reviews the causal-noncausal autoregressive processes and their estimation techniques, with a focus on the GCov estimator. Section 3 studies the behavior of the MAR process and its tail dynamics during a bubble period. Section 4 develops test statistics to detect the bubbles and determine their duration. In Section 5, we apply our approach to investigate the presence and duration of "green bubbles" in the green energy stock market. Section 6 concludes. Appendices A and B contain additional technical results, and Appendix C presents the simulations.

2 Causal-Noncausal Processes

The causal-noncausal models represent stationary processes characterized by locally explosive patterns, such as bubbles and spikes. The univariate causal-noncausal models were examined, for example, by [Breidt et al. \(1991\)](#) and [Lanne and Saikkonen \(2011\)](#), and extended to multivariate analysis by [Lanne and Saikkonen \(2013\)](#), [Gourieroux and Jasiak \(2017\)](#), and [Davis and Song \(2020\)](#). In applied research, causal-noncausal models were used to study various economic and financial variables, including Bitcoin prices [[Hencic and Gouriéroux \(2015\)](#), [Cavaliere et al. \(2020\)](#)], stock market indices [[Gourieroux and Zakoian \(2017\)](#)], commodity prices [[Hecq and Voisin \(2021\)](#), [Lof and Nyberg \(2017\)](#)], and inflation rates [[Lanne and Saikkonen \(2013\)](#), [Hecq and Voisin \(2023\)](#)].

One of the main advantages of these models is their ability to capture complex nonlinear patterns, such as local trends and conditional heteroskedasticity, while still resembling traditional linear time series models in specification. However, the standard Box–Jenkins

approach to identifying and estimating linear time series processes does not apply here, as it relies on the assumption of Gaussian errors. Under this assumption, causal and noncausal dynamics cannot be distinguished (see [Gourieroux and Monfort \(2015\)](#)). Therefore, identification requires assuming non-Gaussian error distributions in causal–noncausal processes.

2.1 Univariate Causal-Noncausal Models

A strictly stationary univariate mixed causal-noncausal autoregressive $MAR(r, s)$ model is defined as:

$$\Phi(L)\Psi(L^{-1})y_t = \epsilon_t, \quad (1)$$

where the error term ϵ_t is non-Gaussian, independent, identically distributed (i.i.d.) and such that $E(|\epsilon_t|^\delta) < \infty$ for $\delta > 0$ [[Lanne and Saikkonen \(2011\)](#)]. The polynomial $\Phi(L)$ in the lag operator L is of the order r . The polynomial $\Psi(L^{-1})$ in the leading operator L^{-1} is of order s . Both polynomials $\Phi(L)$ and $\Psi(L^{-1})$ have roots outside the unit circle.

The $MAR(r, s)$ process (1) admits a unique strictly stationary solution, which is a two-sided moving average of order infinity $MA(\infty)$:

$$y_t = \sum_{h=-\infty}^{\infty} c_h \epsilon_{t-h},$$

in past, present and future shocks, with $c_0 = 1$ [[Breidt et al. \(1991\)](#)]. The coefficients c_h are uniquely defined, provided that (ϵ_t) are non-Gaussian. When y_t is purely noncausal (resp. causal), the coefficients c_h are zero for all $h > 0$ (resp. $h < 0$). Thus, a purely causal process is determined only by the past and present shocks, while a purely noncausal process is influenced only by the present and future shocks. For $r = s = 1$, we obtain the $MAR(1, 1)$ process:

$$(1 - \phi L)(1 - \psi L^{-1})y_t = \epsilon_t, \quad (2)$$

with $|\psi| < 1, |\phi| < 1$, which is purely causal (resp. noncausal) if $\psi = 0$ (resp. $\phi = 0$). For each of these pure processes, the effects of a large ϵ_t are easily distinguished, as a large error leads to a (vertical) jump if $\psi = 0$ and $\phi > 0$, and an explosive bubble with a (vertical) burst if $\psi > 0$ and $\phi = 0$.

The $MAR(1, 1)$ process can be decomposed into the following unobserved components [[Lanne and Saikkonen \(2011\)](#)]:

$$u_t \equiv (1 - \phi L)y_t \text{ or, } (1 - \psi L^{-1})u_t = \epsilon_t, \quad (3)$$

and

$$v_t \equiv (1 - \psi L^{-1})y_t \text{ or, } (1 - \phi L)v_t = \epsilon_t. \quad (4)$$

Gourieroux and Jasiak (2016) shows that u_t is ϵ -noncausal (dependent on the future and present values of ϵ) and y -causal (dependent on the past and present values of y), representing the regular dynamics of y_t . In contrast, v_t is ϵ -causal (dependent on the past and present values of ϵ) and y -noncausal (dependent on the future and present values of y), representing the explosive part of the process, including bubbles and spikes. Thus, u_t, v_t are the unobserved causal and noncausal components of y_t .

The process y_t can be described by the following deterministic representation based on the unobserved components and used e.g. to simulate the MAR processes:

$$y_t = \frac{1}{1 - \phi\psi}(\phi v_{t-1} + u_t), \text{ or } y_t = \frac{1}{1 - \phi\psi}(v_t + \psi u_{t+1}).$$

We observe that y_t is a linear function of the first lag of the ϵ -causal component v_t and of the current value of the ϵ -noncausal component u_t . Alternatively, y_t can be expressed as a linear function of the current value of the ϵ -causal component v_t and of the first lag of the ϵ -noncausal component u_t . The noncausal component v_t represents the explosive part of the process, such as bubbles and spikes. On the other hand, u_t is the purely causal component of the process y_t .

Moreover, the autocovariances of the latent components defined in eq. (3) and (4), respectively, help us distinguish the bubble episode in the process (y_t). We observe that the autocovariances at lag $h \geq 1$ of u_{t+h} and v_t conditional on $y_t = y$ are time-varying, in general, provided that y is not an extreme [see Appendix B]. When y is large, their behavior is different and can be examined using the concept of a tail process described in Section 3. In practice, the latent components are computed given the estimated values of the autoregressive parameters. The estimation methods are discussed in the next section.

Remark 1: An alternative univariate representation of the causal-noncausal process can be written as:

$$y_t = \varphi_1 y_{t-1} - \cdots - \varphi_p y_{t-p} + \varepsilon_t,$$

where $\varepsilon_t = -\frac{1}{\psi_s} \epsilon_t$ [Brockwell and Davis (1987)]. In this representation, the polynomial $\Phi(L) = 1 - \varphi_1 L - \cdots - \varphi_p L^p$ has roots both inside and outside the unit circle. Specifically, the roots outside the unit circle correspond to the causal component of y_t , while the roots inside the unit circle indicate the noncausal component that captures bubbles and other nonlinear features. Note that in either representation of the model, the error ε_t is not an innovation process because ε_t is not independent of $y_{t-1}, y_{t-2}, \dots$.

2.2 The GCov Estimator

One way to estimate and identify $\text{MAR}(r, s)$ models is by using a parametric non-Gaussian Maximum Likelihood (ML) approach (see, e.g. [Hecq et al. \(2016\)](#)). Alternatively, the semi-parametric Generalized Covariance (GCov) estimator can be used, which does not require any distributional assumptions on the errors, other than non-Gaussianity for the identification of parameters. It is a one-step estimator that is consistent, asymptotically normally distributed, and semi-parametrically efficient. It can achieve parametric efficiency in special cases [[Gourieroux and Jasiak \(2023\)](#)]. The GCov minimizes a portmanteau-type objective function involving nonlinear autocovariances, that is, the autocovariances of nonlinear transformations of model errors, which successfully identify the causal and non-causal dynamics [[Chan et al. \(2006\)](#)].

For example, let us consider the nonlinear functions $a(\varepsilon_t) = [a_1(\varepsilon_t)', \dots, a_K(\varepsilon_t)']'$ that satisfy the regularity conditions given in [Gourieroux and Jasiak \(2023\)](#). This increases the dimension of the process from 1 to K . Let us denote by $\hat{\Gamma}^a(h; \theta)$, $h = 1, \dots, H$ the autocovariance matrices at lags $h = 0, \dots, H$, with $\hat{\Gamma}^a(0; \theta)$ representing the variance, and θ the vector of autoregressive parameters. Then, the GCov estimator of the parameter $\hat{\theta}_T$ minimizes the following objective function:

$$\hat{\theta}_T = \underset{\theta}{\operatorname{argmin}} \sum_{h=1}^H \operatorname{Tr} \left[\hat{\Gamma}^a(h; \theta) \hat{\Gamma}^a(0; \theta)^{-1} \hat{\Gamma}^a(h; \theta)' \hat{\Gamma}^a(0; \theta)^{-1} \right], \quad (5)$$

where Tr denotes the trace of a matrix and we denote $L_T(\hat{\theta}_T, H)$ the value of the function at its minimum. Alternatively, when the number of transformations K is large, we can replace in the above formula $\hat{\Gamma}^a(0; \theta)$ with $\operatorname{diag}(\hat{\Gamma}_a(0; \theta))$, which is the variance matrix containing only the diagonal elements of $\hat{\Gamma}_a(0; \theta)$. This latter version of the GCov estimator is no longer semiparametrically efficient, as it is not optimally weighted [see [Gourieroux and Jasiak \(2023\)](#), [Cubadda and Hecq \(2011\)](#)]. Another strategy for the diagonal GCov is the regularized GCov proposed by [Giancaterini et al. \(2025\)](#).

The choice of an informative set of transformations $(a_k, k = 1, \dots, K)$ depends on the specific series under investigation. For example, in financial applications, linear and quadratic functions can be selected, such as $a_1(\varepsilon_t) = \varepsilon_t$, $a_2(\varepsilon_t) = \varepsilon_t^2$. This implies that a_1 is a linear function of errors in a causal-noncausal process, while a_2 transforms the error term by squaring it for each $t = 1, \dots, T$, where T represents the total number of observations. In application to MAR processes with heavy-tailed error distributions, such as stable distributions, including Cauchy, or t-student distribution with low degrees of freedom, one can

use the square root of absolute values or other fractional powers to ensure the asymptotic normality of the GCov estimator.

Moreover, the objective function minimized in (5) and evaluated at the estimated parameter $\hat{\boldsymbol{\theta}}_T$ can be used to test the fit of the model. Specifically, we test the null hypothesis:

$$H_0 : \{\Gamma_0^a(h) = 0, h = 1, \dots, H\},$$

using the statistic

$$\hat{\xi}_T(H) = TL_T(\hat{\boldsymbol{\theta}}_T, H),$$

which, under the implicit null hypothesis of serial independence of errors, has an asymptotic chi-square distribution with degrees of freedom equal to $HK^2 - \dim(\boldsymbol{\theta})$ [Gourieroux and Jasiak (2023)].

3 Bubble Analysis

Our approach to bubble analysis assumes that the process of interest follows a strictly stationary MAR(r, s) process with errors from a heavy-tailed distribution with tail parameter $\alpha \in (0, 2)$ that includes the Cauchy distribution for $\alpha = 1$, for example, or a t-student distribution with a low number of degrees of freedom. In this Section, we focus our attention on MAR processes of combined orders r and s less than or equal to 2, so that the process of interest is either a MAR(0,1), i.e., a purely noncausal process of order 1, or a MAR(1,1), or a causal MAR(1,0). These processes are geometrically ergodic. [Gourieroux and Zakoian \(2017\)](#) and [Fries and Zakoian \(2019\)](#) also show that the MAR(0,1) and MAR(1,1) processes are Markov of order 1 and 2, respectively. The results can be generalized to higher-order MAR(r, s) processes, which are Markov too [[Fries and Zakoian \(2019\)](#), Proposition 3.1].

3.1 On-Bubble Dynamics

Consider the locally explosive MAR(0,1) and MAR(1,1) processes. It has been shown in the literature that during a bubble episode, and conditional on a large $y_t > y$, the causal-noncausal MAR(0,1) and MAR(1,1) processes with α -stable distributed errors, $\alpha \in (0, 2)$ have distinct dynamics [[Fries \(2022\)](#)], which leads to different behavior of conditional autocovariances of its latent components. In this Section, we describe these dynamics and generalize the results in [Fries \(2022\)](#) using the concept of a spectral tail process, based only on observations above a high threshold. We have the following result:

Proposition 1 [Kulik and Soulier (2020), Chapter 15.3]: Let $\{y_t, t \in \mathbb{Z}\}$ be a strictly stationary process with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with (Pareto-type) tails of tail index α and with a two-sided MA representation:

$$y_t = \sum_{h=-\infty}^{+\infty} c_h \epsilon_{t-h},$$

with nonnegative coefficients. The process $\left(\frac{y_{t+h}}{y_t}\right)_h, h \in \mathbb{Z}$, converges weakly to the (spectral) tail process (X_h) , so that conditional on a large $y_t > y$, where y is a high threshold, we have :

$$\mathcal{L}\left(\frac{y_{t+h}}{y_t}, h = -H, \dots, H | y_t > y\right) \xrightarrow{w} \mathcal{L}(X_h, h = -H, \dots, H),$$

where $\xrightarrow{w}$ denotes weak convergence, also denoted as $\frac{y_{t+h}}{y_t} \xrightarrow{d} X_h$ for $h \in \mathbb{Z}$, and

$$X_h = \frac{c_{h+N}}{c_N} X_0, \quad h \in \mathbb{Z}, \quad (6)$$

where $X_0 = 1$. N is an integer-valued random variable, independent of y and such that:

$$P[N = h] = \frac{c_h^\alpha}{\sum_{h \in \mathbb{Z}} c_h^\alpha}, \quad \forall h \in \mathbb{Z}. \quad (7)$$

Thus, setting $X_0 = 1$, we consider an upward (positive) bubble. Henceforth, we focus our attention on this case for ease of exposition. Similar results are easily obtained for downward bubbles, conditional on $y_t < y$ where y tends to $-\infty$ with $X_0 = -1$. This result can be applied to the MAR(1,1) and MAR(0,1) processes as follows. Let us first consider the MAR(1,1) process defined in Section 2.1, assuming positive coefficients ϕ and ψ .

Proposition 2: The strictly stationary MAR(1,1) process with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with tail index α :

$$(1 - \phi L)(1 - \psi L^{-1})y_t = \epsilon_t,$$

- (i) admits a two-sided MA(∞) representation with the coefficients: $c_h = \frac{1}{1 - \phi\psi} \psi^{-h}$, if $h \leq 0$, and $c_h = \frac{1}{1 - \phi\psi} \phi^h$, if $h \geq 0$.
- (ii) the tail process X_h is such that:

$$X_h = \psi^{-1} X_{h-1} \mathbb{1}_{N \leq -h} + \phi X_{h-1} \mathbb{1}_{N > -h},$$

with $X_0 = 1$, and the probability $P[N = h]$:

$$P[N = h] = \frac{\psi^{-h\alpha}}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]}, \text{ if } h \leq 0, \text{ and } P[N = h] = \frac{\phi^{h\alpha}}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]}, \text{ if } h \geq 0,$$

with $0^0 = 1$, by convention.

Proof: See Appendices A1-A2.

We observe that the distribution of variable N is a mixture of two geometric distributions (see Proposition 4).

Corollary 1: In the MAR(1,1) process, we have:

$$X_h = \phi^h \mathbb{1}_{N > \max(-h, 0)} + \phi^{-h-N} \psi^{-N} \mathbb{1}_{0 < N \leq -h} + \phi^{h+N} \psi^N \mathbb{1}_{-h < N \leq 0} + \psi^{-h} \mathbb{1}_{N \leq \min(-h, 0)}$$

Proof: We proceed in three steps. First, we solve the backward and forward recursive equations satisfied by the sequences X_h starting from the same initial (terminal) condition X_{-N} . Next, we determine the value of X_{-N} given the known value $X_0 = 1$. In the last step, we combine the results.

(i) From Proposition 2 it follows that for $N > -h$ we have :

$$X_h = \phi X_{h-1}$$

Since $N > -h \iff h > -N \iff h \geq -N + 1$, this leads to a recursive formula with the initial condition X_{-N} and the general term

$$X_h = \phi^{h+N} X_{-N}, \text{ for } h \geq -N + 1$$

(ii) It follows from Proposition 2 that for $N \leq -h$, we have:

$$X_h = \frac{1}{\psi} X_{h-1},$$

which can be written as:

$$X_{h-1} = \psi X_h.$$

Since $N \leq -h \iff -h \geq N \iff h \leq -N$, this leads to a recursive formula with the terminal condition X_{-N} and the general term:

$$X_h = \psi^{-h-N} X_{-N}$$

(iii) These results can be combined, yielding:

$$X_h = (\phi^{h+N} \mathbb{1}_{N > -h} + \psi^{-h-N} \mathbb{1}_{N \leq -h}) X_{-N}$$

(iv) The value of X_{-N} is unknown and can be determined from $X_0 = 1$ by evaluating the above formula for $h = 0$. We get:

$$\begin{aligned} 1 = X_0 &= (\phi^N \mathbb{1}_{N>0} + \psi^{-N} \mathbb{1}_{N\leq 0}) X_{-N} \\ \iff X_{-N} &= \phi^{-N} \mathbb{1}_{N>0} + \psi^N \mathbb{1}_{N\leq 0} \end{aligned}$$

(v) Next, we compute:

$$\begin{aligned} X_h &= (\phi^{h+N} \mathbb{1}_{N>-h} + \psi^{-h-N} \mathbb{1}_{N\leq -h}) (\phi^{-N} \mathbb{1}_{N>0} + \psi^N \mathbb{1}_{N\leq 0}) \\ &= \phi^h \mathbb{1}_{N>-h} \mathbb{1}_{N>0} + \psi^{-h-N} \phi^{-N} \mathbb{1}_{N\leq -h} \mathbb{1}_{N>0} + \phi^{h+N} \psi^N \mathbb{1}_{N>-h} \mathbb{1}_{N\leq 0} + \psi^{-h} \mathbb{1}_{N\leq -h} \mathbb{1}_{N\leq 0} \\ &= \phi^h \mathbb{1}_{N>\max(-h,0)} + \psi^{-h-N} \phi^{-N} \mathbb{1}_{0<N\leq -h} + \phi^{h+N} \psi^N \mathbb{1}_{-h<N\leq 0} + \psi^{-h} \mathbb{1}_{N\leq \min(-h,0)} \end{aligned}$$

During a bubble episode, we can distinguish the phases of growth and decline. The noncausal persistence of MAR(1,1) processes determines the rate at which a bubble keeps growing up to time $t-N$. The negative power of ψ indicates that the growth is explosive while $N < 0$ given that $|\psi| < 1$ is assumed for strict stationarity. The above result is consistent with Proposition 4.2 in [Fries \(2022\)](#) which describes the dynamics of a MAR(1,1) process conditional on a large value $y_t > y$ and $\frac{y_{t+1}}{y_t} = \frac{y_{t+2}}{y_{t+1}} = 1/\psi$. In the MAR(1,1), conditional on $y_t > y$, the bubble either keeps growing to the next value $y_{t+1} = \frac{1}{\psi} y_t$, or it bursts and decreases to $y_{t+1} = \phi y_t$. This latter pattern is also consistent with formula (v) for $N > 0$, which additionally implies that the bubble bursts at $N = 0$.

In a MAR(0,1), the bubble bursts vertically to 0, while in the causal AR(1), i.e., MAR(1,0), we observe a jump, followed by a decrease determined by the autoregressive coefficient, as shown below.

Corollary 2: The strictly stationary MAR(0,1) process with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with tail index α :

$$(1 - \psi L^{-1})y_t = y_t - \psi y_{t+1} = \epsilon_t,$$

admits a one sided MA(∞) representation with the moving average coefficients:

$$c_h = \psi^{-h}, \text{ if } h \leq 0, \text{ and } c_h = 0, \text{ if } h > 0.$$

The tail process is such that:

$$X_h = \psi^{-1} X_{h-1} \mathbb{1}_{N \leq -h},$$

with $X_0 = 1$ and the probability $P[N = h]$:

$P[N = h] = 0$, if $h > 0$, and $P[N = h] = (1 - \psi^\alpha)\psi^{-h\alpha}$, if $h \leq 0$,

It follows from Proposition 2 that the variable $-N$ has a geometric distribution on $\mathbb{N}$ with parameter ψ^α during the growth phase of bubble.

The causal autoregressive processes with errors from a heavy-tailed distribution with tail index α may admit a tail process behavior following a jump.

Corollary 3: The strictly autoregressive of order 1 (causal AR(1), i.e. MAR(1,0)) process:

$$y_t = \phi y_{t-1} + \epsilon_t,$$

with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with tail index α and $0 < \phi < 1$ admits a one sided MA(∞) with coefficients:

$$c_h = \phi^h \text{ if } h \geq 0, \text{ and } c_h = 0, \text{ if } h < 0.$$

The tail process is such that:

$$X_h = \phi X_{h-1} \mathbb{1}_{N \geq h},$$

with $X_0 = 1$ and the probability $P[N = h]$:

$$P[N = h] = 0; \text{ if } h < 0, \text{ and } P[N = h] = (1 - \phi^\alpha)\phi^{h\alpha}; \text{ if } h \geq 0,$$

Hence, the path of y_t displays jumps, with a geometric decline. The variable N has a geometric distribution with parameter ϕ^α during the decline following a jump. Moreover, the behavior of pure noncausal autoregressive processes of order 2 (MAR(0,2) with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with tail index α is described in Appendix B.

The results given above suggest the following definition of a bubble:

Definition: The bubble is an episode when the observations above a high threshold on a strictly stationary autoregressive noncausal process with i.i.d. errors from a heavy-tailed distribution with tail index α increase (decline) at the rate of a tail process.

From Proposition 1, it follows that at any time t when $y_t > y$ and y is a high threshold, the ratios of observations $y_{t+h}/y_t \stackrel{d}{\approx} X_h$ for any h , where $\stackrel{d}{\approx}$ denotes an approximate equality of distributions.

3.2 Tail Behavior of Latent Components

The latent components are defined in equations (3) and (4) as:

$$u_{t+1} = (1 - \phi L)y_{t+1} = y_{t+1} - \phi y_t \text{ and } v_t = (1 - \psi L^{-1})y_t = y_t - \psi y_{t+1}, t = 1, \dots$$

for the MAR(1,1) process. We observe that the ratios of these latent components divided by y_t :

$$\frac{u_{t+h}}{y_t} = \frac{y_{t+h} - \phi y_{t+h-1}}{y_t} \text{ and } \frac{v_{t+h}}{y_t} = \frac{y_{t+h} - \psi y_{t+h+1}}{y_t}$$

are functions of ratios y_{t+h}/y_t for $h \in -H, \dots, H$. Our goal is to replace them by the tail components: $U_h = X_h - \phi X_{h-1}$ and $V_h = X_h - \psi X_{h+1}$, respectively. Then, when $y_t > y$ is large, they are linear deterministic functions of the components X_h of the tail process.

Let us now examine how Proposition 2 can be used to obtain the asymptotic behavior (in distribution) of U_h, V_h , $h = -H, \dots, H$.

Proposition 3: At time t such that $y_t > y$, for a high threshold y , we have:

$$u_{t+h}/y_t \stackrel{d}{\approx} X_h - \phi X_{h-1} \equiv U_h, \quad v_{t+h}/y_t \stackrel{d}{\approx} X_h - \psi X_{h+1} \equiv V_h,$$

and

$$U_h = (\psi^{-1} - \phi)X_{h-1}\mathbb{1}_{N \leq -h}, \quad V_h = (1 - \psi\phi)X_h\mathbb{1}_{N > -h-1}.$$

Proof: Let us consider the causal component. It follows from Proposition 2 that:

$$U_h = X_h - \phi X_{h-1} = (\psi^{-1} - \phi)X_{h-1}\mathbb{1}_{N \leq -h}.$$

Similarly, we have:

$$V_h = X_h - \psi X_{h+1} = X_h - X_h\mathbb{1}_{N \leq -h-1} - \psi\phi X_h\mathbb{1}_{N > -h-1} = (1 - \phi\psi)X_h\mathbb{1}_{N > -h-1}.$$

Corollary 4: The causal and noncausal tail components U_h, V_h are such that:

$$\begin{aligned} U_h &= 0, \text{ if } N > -h \iff t + h > t - N, \\ V_h &= 0, \text{ if } N \leq -h - 1 \iff t + h < t - N - 1, \end{aligned}$$

Proposition 3 and Corollary 4 can be interpreted as follows. Let us consider an exogenous time t when $y_t > y$, for large y . Then t belongs to a bubble episode, with a peak at time $t - N$. The noncausal (resp. causal) tail component is equal to zero after (resp. before) the peak, and has the behavior of a MAR(0,1) tail process before the peak (resp. MAR(1,0) tail process after the peak). This result can also be used to determine the behavior of products of the causal and noncausal tail components.

Corollary 5: We have

$$\xi_{t,h,k} = \frac{u_{t+h}v_{t+k}}{y_t^2} \stackrel{d}{\approx} U_h V_k = 0, \text{ if } h - k \geq 1.$$

Proof: We see that $U_h V_k = 0 \iff (N > -h) \text{ or } (N \leq -k-1)$. This condition is equivalent to $N \in \mathbb{Z}$ that is always satisfied iff:

$$\begin{aligned}\mathbb{Z} &= (-\infty, -k-1] \cup [-h+1, \infty] \\ \iff -h+1 &\leq -k-1+1 = -k \\ \iff h-k &\geq 1.\end{aligned}$$

The result follows. In particular, we have:

$$\xi_{t,h} = \frac{u_{t+h+1}v_{t+h}}{y_t^p} \stackrel{d}{\approx} U_{h+1}V_h = 0, \forall h$$

where $p = r + s = 2$ is the combined autoregressive order of the process, and

$$\xi_{t,0} = \frac{u_{t+1}v_t}{y_t^p} \stackrel{d}{\approx} U_1V_0 = 0.$$

For the MAR(0,1) process with $p = 1$, we get:

(i) at $h = 0$.

$$\xi_{t,0} = \frac{v_t}{y_t} \stackrel{d}{\approx} 1 - \psi X_1,$$

where

$$X_1 = \psi^{-1} \mathbb{1}_{N \leq -1}.$$

The statistic is: $\xi_{t,0} = 1 - \mathbb{1}_{N \leq -1}$ and is always 0 during the bubble.

(ii) at $h = 1$.

$$\xi_{t,1} = \frac{v_{t+1}}{y_t} \stackrel{d}{\approx} X_1 - \psi X_2$$

where

$$X_2 = \psi^{-1} X_1 \mathbb{1}_{N \leq -2}$$

The statistic $\xi_{t,1}$ is equal to 0 as well.

The observed behavior of statistic $\xi_{t,h}$ can be used for testing the hypothesis that the process is in a bubble at time $t + h$. This approach is pursued in Section 4. Below, we discuss the behavior of the variable N and associated inference.

3.3 Time to Peak

This section describes the stochastic properties of N , which determines the duration of a bubble, and of $t - N$ for $N \leq 0$, which is interpreted as the time to peak. We consider first the MAR(1,1) process.

Proposition 4: In the MAR(1,1) process, the distribution of N is a mixture of two geometric distributions with $P[N \geq 0] = \frac{1}{1 - \phi^\alpha} / (\frac{1}{1 - \phi^\alpha} + \frac{\psi^\alpha}{1 - \psi^\alpha})$. The distribution of N given $N \geq 0$ is a geometric distribution in $\mathbb{N}$ with parameter ϕ^α , and the distribution of $-1 - N$, given $N \leq -1$ is a geometric distribution in $\mathbb{N}$ with parameter ψ^α .

Proof: From Proposition 2, it follows that:

$$P[N \geq 0] = \frac{1}{1 - \phi^\alpha} / \left(\frac{1}{1 - \phi^\alpha} + \frac{\psi^\alpha}{1 - \psi^\alpha} \right).$$

Then, for $h \geq 0$:

$$P(N = h | N \geq 0) = \frac{\phi^{h\alpha}}{1 - \phi^\alpha}, \text{ which is a geometric distribution in } \mathbb{N} \text{ with the parameter } \phi^\alpha.$$

For $h < 0$, we have:

$$P(N = h | N < 0) = \frac{\psi^{(-h-1)\alpha}}{1 - \psi^\alpha}, \text{ which means that } -1 - N \text{ has a geometric distribution on } \mathbb{N} \text{ with parameter } \psi^\alpha.$$

Corollary 6: The expected value of N is:

$$E(N) = \frac{1}{\frac{1}{1 - \phi^\alpha} + \frac{1}{1 - \psi^\alpha} - 1} \left(\frac{\phi^\alpha}{(1 - \phi^\alpha)^2} - \frac{\psi^\alpha}{(1 - \psi^\alpha)^2} \right).$$

Proof: We have

$$\begin{aligned} E(N) &= E(N | N \geq 0)P(N \geq 0) + E(N | N \leq -1)P(N \leq -1) \\ &= E(Z_1)P(N \geq 0) + E(-1 - Z_2)P(N \leq -1), \end{aligned}$$

where Z_1, Z_2 follow geometric distributions on $\mathbb{N}$ with parameters ϕ^α and ψ^α , respectively. It follows that:

$$\begin{aligned} E(N) &= \frac{1}{\frac{1}{1 - \phi^\alpha} + \frac{\psi^\alpha}{1 - \psi^\alpha}} \left[\frac{1}{1 - \phi^\alpha} \frac{\phi^\alpha}{1 - \phi^\alpha} + \frac{\psi^\alpha}{1 - \psi^\alpha} \left(-1 - \frac{\psi^\alpha}{1 - \psi^\alpha} \right) \right] \\ &= \frac{1}{\frac{1}{1 - \phi^\alpha} + \frac{\psi^\alpha}{1 - \psi^\alpha}} \left(\frac{\phi^\alpha}{(1 - \phi^\alpha)^2} - \frac{\psi^\alpha}{(1 - \psi^\alpha)^2} \right). \end{aligned}$$

The marginal expected value of N depends on the values of ϕ and ψ and is symmetric in ϕ and ψ . In particular, it is equal to 0 if $\phi = \psi$, which corresponds to a symmetric bubble. In addition, this expectation depends on α . The effects of ϕ, ψ, α are illustrated in Figure 1, where we observe that, for a fixed ψ , the expected value $E(N)$ increases in ϕ . For a fixed ϕ , $E(N)$ increases in absolute values in ψ . In addition, the range of values of $E(N)$ depends on the tail parameter α .

If the bubble is asymmetric, then $E(N)$ can take either a positive or negative value. Therefore, one may be interested in computing the conditional expectation of N given $N < 0$ to approximate the expected time to peak.

It is also interesting to compute the cumulative distribution function (cdf), or survival function of N , to derive the quantiles of the distribution of N and obtain the prediction interval for N . We find the quantiles h_U^* and h_L^* of the distribution of N at levels $(1 - \gamma)$ and γ , where γ is small, which are: h_U^* such that $P[N \leq h_U^*] = \gamma$, and h_L^* such that $P[N > h_L^*] = 1 - \gamma$. Using small γ , we separately consider the geometric distributions of the mixture.

Corollary 7: For $h \leq 0$, sufficiently small, we have:

$$P[N \leq h] = \frac{\psi^{-h\alpha}}{1 - \psi^\alpha} \frac{1}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]},$$

and for $h > 0$, sufficiently large, we have:

$$P[N > h] = \frac{\phi^{h\alpha}}{1 - \phi^\alpha} \frac{1}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]}.$$

Proof: see Appendix.

The expected value (median or mode) provides a point prediction of the N . The quantiles of the distribution of N provide the prediction interval for N .

For the MAR(0,1) process, the variable $-N$ follows a geometric distribution on $\mathbb{N}$ with parameter ψ^α (by Corollary 2 or Proposition 4) during bubble growth. The cumulative probability function of N is:

$$P[N \leq h] = (1 - \psi^\alpha) \sum_{i \leq h} \psi^{-i\alpha} = (1 - \psi^\alpha) \sum_{i \geq -h} \psi^{i\alpha} = \psi^{-h\alpha},$$

for $h \leq 0$. The mean of this geometric distribution is equal to

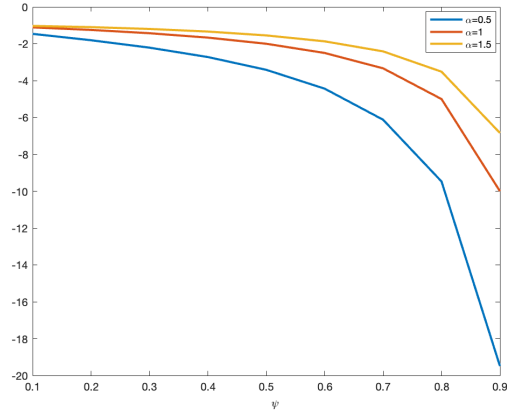
$$E(N) = -\frac{1}{1 - \psi^\alpha},$$

and can be interpreted as the average time to peak. The median is:

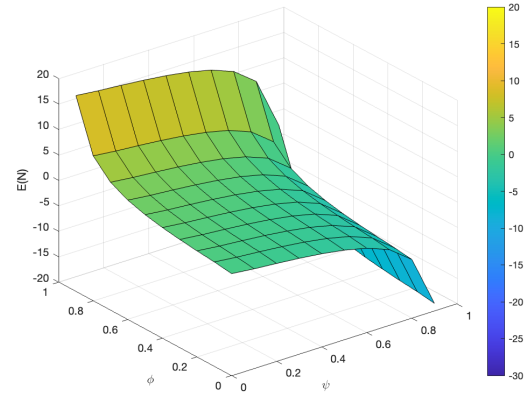
$$med(N) = \left\lceil \frac{\log 2}{\alpha \log(\psi)} \right\rceil,$$

or its integer part, and the mode is equal to $mode(N) = 0$. Since the distribution of $-N$ is skewed, the expectation, median, and mode provide three different point predictions. We illustrate the effect of the parameters ψ and α in Figure 1a. We observe that for MAR(0,1),

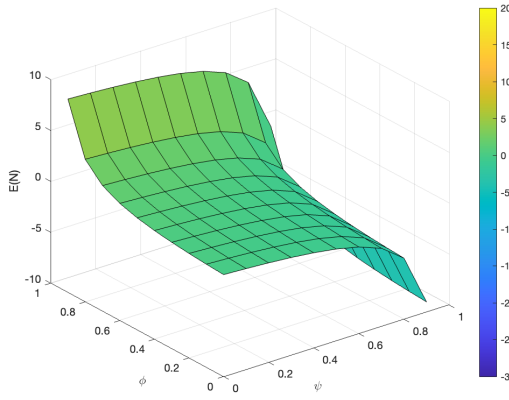
Figure 1: $E(N)$ for $MAR(0,1)$ and $MAR(1,1)$



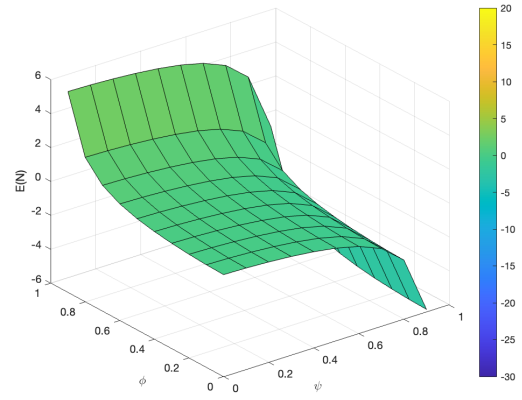
(a) $E(N)$ for $MAR(0,1)$



(b) $E(N)$ for $MAR(1,1)$, $\alpha = 0.5$



(c) $E(N)$ for $MAR(1,1)$, $\alpha = 1$



(d) $E(N)$ for $MAR(1,1)$, $\alpha = 1.5$

the absolute value of the expected time to peak increases in ψ . This is consistent with the dynamics of this process, with a growing phase of a bubble followed by an instantaneous crash. Moreover, the values of $E(N)$ depend on the parameter α .

Up to the effect of the integer part, both the median and the expectation decrease in ψ^α . To build the prediction interval, we use the quantiles of the geometric distribution, which are determined by inverting the cumulative probability function $P[N \leq h] = \psi^{-h\alpha}$. For a level γ , the quantile is equal to :

$$h(\gamma) = -\frac{\log \gamma}{\alpha \log \psi}$$

and the prediction interval for N at level 10% is:

$$\left(-\frac{\log 0.05}{\alpha \log \psi}, -\frac{\log 0.95}{\alpha \log \psi}\right).$$

Because the variable is discrete, we round up or down the upper and lower quantiles to integer values to make it more conservative.

In practice, the predictions and prediction intervals of N can be found by replacing the unknown parameters ϕ, ψ by their GCov-based estimates based on a sample of T observations, which are also used for computing the test statistics given in the next section. The tail parameter α can be approximated by the Hill estimator, which suffers from the bias-variance trade-off in finite samples. [Kulik and Soulier \(2020\)](#) shows that under the tail process approach, the Hill estimator is consistent and asymptotically normally distributed.

4 Inference

4.1 The Statistics

Let us show how the results derived in Section 3.2 can be used to build diagnostic tools to test the MAR(1,1) process for bubbles. Diagnostics are based on the counterparts of

$$\xi_{t,0} = \frac{u_{t+1}v_t}{y_t^2} = \left(\frac{y_{t+1}}{y_t} - \phi_0\right) \left(1 - \psi_0 \frac{y_{t+1}}{y_t}\right),$$

where ϕ_0, ψ_0 denote the true values of the parameters. These quantities depend on the observations and the unknown parameters. The unknown parameters can be replaced by consistent and asymptotically normally distributed estimators $\hat{\phi}_T, \hat{\psi}_T$ of ϕ, ψ obtained from the sample of T observations. Then, the sample counterpart of $\xi_{t,0}$ is:

$$\hat{\xi}_{t,T} = \left(\frac{y_{t+1}}{y_t} - \hat{\phi}_T\right) \left(1 - \hat{\psi}_T \frac{y_{t+1}}{y_t}\right).$$

The statistics $\hat{\xi}_{t,T}, t = 1, \dots, T$ can be used as follows. From Corollary 5, we know that if y_t is sufficiently large at time t , then $\xi_t \stackrel{d}{\approx} U_1 V_0$ and that $U_1 V_0 = 0$. Therefore, we expect that $\hat{\xi}_{t,T}$ is close to 0. We can also consider another lag h with:

$$\xi_t(h) = \frac{u_{t+h+1}v_{t+h}}{y_t^2} = \left(\frac{y_{t+h+1}}{y_t} - \phi_0 \frac{y_{t+h}}{y_t} \right) \left(\frac{y_{t+h}}{y_t} - \psi_0 \frac{y_{t+h+1}}{y_t} \right).$$

It was shown in Corollary 5 that $\xi_{t,h} \stackrel{d}{\approx} U_{h+1} V_h$ and $U_{h+1} V_h = 0$. The sample counterpart is

$$\hat{\xi}_{t,T}(h) = \frac{\hat{u}_{t+h+1}\hat{v}_{t+h}}{y_t^2} = \left(\frac{y_{t+h+1}}{y_t} - \hat{\phi}_T \frac{y_{t+h}}{y_t} \right) \left(\frac{y_{t+h}}{y_t} - \hat{\psi}_T \frac{y_{t+h+1}}{y_t} \right).$$

4.2 Confidence Band

Consider the statistic $\hat{\xi}_{t,T}$. The zero value of the transformed tail process can be considered as the true value of some tail parameter $\theta_1 = U_1 V_0$, which is deterministic and equal to 0 if y_t is sufficiently large, and it is stochastic, otherwise. Then, at each exogenous time t , we can consider the null hypothesis:

$$H_{0,1} = \{\theta_1 = 0\}.$$

The difficulty is that we have a double asymptotic in the level of threshold y and in the number of observations T . To derive reliable confidence bands, we assume that during the bubble episode, the uncertainty in the approximation $\xi_{t,0} \stackrel{d}{\approx} U_1 V_0$ is negligible with respect to the asymptotics in T . Then, during a bubble episode when $H_{0,1}$ is satisfied and $y_t > y$, with large y , we have conditional on y_t, y_{t+1} :

$$\begin{aligned} \sqrt{T}(\hat{\xi}_{t,T} - \xi_{t,0}) &\stackrel{d}{=} \sqrt{T}(\hat{\xi}_{t,T} - U_1 V_0) \\ &= \sqrt{T}\hat{\xi}_{t,T} \xrightarrow{d} N(0, \sigma_t^2), \end{aligned}$$

where σ_t^2 is obtained by the delta method as follows:

$$\sigma_t^2 = V_t \left\{ \left[- \left(1 - \psi_0 \frac{y_{t+1}}{y_t} \right), \frac{-y_{t+1}}{y_t} \left(\frac{y_{t+1}}{y_t} - \phi_0 \right) \right] \sqrt{T} \begin{pmatrix} \hat{\phi}_T - \phi_0 \\ \hat{\psi}_T - \psi_0 \end{pmatrix} \right\}.$$

This quantity is then consistently estimated from:

$$\hat{\sigma}_{t,T}^2 = \left(\frac{\hat{v}_t}{y_t}, \frac{y_{t+1}\hat{u}_{t+1}}{y_t^2} \right) \hat{\Omega}_T \begin{pmatrix} \frac{\hat{v}_t}{y_t} \\ \frac{y_{t+1}\hat{u}_{t+1}}{y_t^2} \end{pmatrix},$$

where $\hat{\Omega}_T$ is a consistent estimator of the asymptotic variance matrix of the parameter estimators. Then, under the assumption of a MAR(1,1) process and following a large y_t , we

have

$$|\sqrt{T} \hat{\xi}_{t,T} / \hat{\sigma}_{t,T}| \leq 1.96,$$

with an asymptotic probability of 95%. This leads to a functional diagnostic tool that consists of reporting for any time t the quantities $\hat{\xi}_{t,T}$, $t = 1, \dots, T$ along with the band

$$\left(\hat{\xi}_{t,T} \pm 1.96 \frac{\hat{\sigma}_{t,T}}{\sqrt{T}} \right).$$

Since the tail process does not depend on the level of y_t and the values of y_{t+1} , the width of the band is independent of time t , $t = 1, \dots, T$. Then, the times at which the statistic is inside the band are the times associated with a bubble with probability 95%.

This approach is easily extended to other lags. From the statistic $\hat{\xi}_{t,T}(h)$, we can test the null hypothesis:

$$H_{0,h} = \{\theta_h = 0\}$$

where $\theta_h = U_{h+1}V_h$ and $y_t > y$. Then, conditional on y_t, y_{t+h}, y_{t+h+1} we have:

$$\begin{aligned} \sqrt{T}(\hat{\xi}_{t,T}(h) - \xi_{t,h}) &\stackrel{d}{=} \sqrt{T}(\hat{\xi}_{t,T}(h) - U_{h+1}V_h) \\ &= \sqrt{T}\hat{\xi}_{t,T}(h) \xrightarrow{d} N(0, \sigma_t^2(h)) \end{aligned}$$

where $\sigma_t^2(h)$ is:

$$\sigma_t^2(h) = V_t \left\{ \left[\frac{-y_{t+h}}{y_t} \left(\frac{y_{t+h}}{y_t} - \psi_0 \frac{y_{t+h+1}}{y_t} \right), \frac{-y_{t+h+1}}{y_t} \left(\frac{y_{t+h+1}}{y_t} - \phi_0 \frac{y_{t+h}}{y_t} \right) \right] \sqrt{T} \begin{pmatrix} \hat{\phi}_T - \phi_0 \\ \hat{\psi}_T - \psi_0 \end{pmatrix} \right\}.$$

This quantity is then consistently estimated from:

$$\hat{\sigma}_{t,T}^2(h) = \left(\frac{y_{t+h}\hat{v}_{t+h}}{y_t^2}, \frac{y_{t+h+1}\hat{u}_{t+h+1}}{y_t^2} \right) \hat{\Omega}_T \begin{pmatrix} \frac{y_{t+h}\hat{v}_{t+h}}{y_t^2} \\ \frac{y_{t+h+1}\hat{u}_{t+h+1}}{y_t^2} \end{pmatrix}.$$

Under the assumption of a MAR(1,1) process and following a large $y_t > y$:

$$|\sqrt{T} \hat{\xi}_{t,T}(h) / \hat{\sigma}_{t,T}(h)| \leq 1.96,$$

with the asymptotic probability of 95 %. This leads to a set of functional diagnostic tools, where for any time t and h the quantities $\hat{\xi}_{t,T}(h)$, $t = 1, \dots, T$ are reported along with the band:

$$\left(\hat{\xi}_{t,T}(h) \pm 1.96 \frac{\hat{\sigma}_{t,T}(h)}{\sqrt{T}} \right).$$

In practice, the above procedure may not distinguish between a bubble and a short-lasting spike with the same growth rate. In addition, there can be times t during the bubble when the statistic is outside the band because either (i) the MAR(1,1) model is misspecified, or (ii) the MAR(1,1) is well specified, but the value of y_t is not sufficiently large.

This method easily provides the test statistics for the MAR(0,1) and MAR(1,0) processes, based on Corollary 5, with $\hat{\xi}_{t,T} = \hat{v}_t/y_t$, and $\hat{\xi}_{t,T} = \hat{u}_{t+1}/y_t$, respectively. The above results applied to MAR(1,0) processes provide a tool for jump detection in the causal autoregressive process. In each case, the confidence bands are independent of the tail parameter α .

$\hat{\xi}_{t,T}$ can be used as a bubble detection tool in the MAR(1,1) and MAR(0,1) processes, as it is constant and close to 0 during the bubble growth and decline periods, and it is time-varying otherwise. Hence, the first difference $\Delta\hat{\xi}_{t,T}$ is also constant and close to 0 during the bubble growth and decline periods.

Consider a sequence of $\hat{\xi}_{t,T}$, evaluated at $t = 1, 2, \dots$ following a high threshold value y_t . A close to zero value of $\hat{\xi}_{t+1,T}$ following a large y_t is a warning of an upcoming bubble. Since this statistic remains close to 0 throughout the duration of the bubble, the number of values of $\xi_t(1)$, for $t = t+1, t+2, \dots$ that are not statistically significant, is a measure of the duration of the bubble. The first time t_j when $\hat{\xi}_{t_j,T} \approx 0$ marks the start of the bubble. The last time t_J , such that $\hat{\xi}_{t_J,T}(1) \approx 0$ marks the end of the bubble.

4.3 Illustration

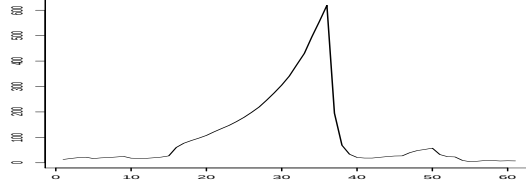
Our approach is illustrated in Figure 2 by a bubble episode of a simulated MAR(1,1) process with $\psi = 0.9, \phi = 0.3$ and Cauchy distributed errors with scale coefficient 1.

The top panel of the Figure displays the bubble episode of the MAR(1,1) process. The first difference $\Delta\hat{\xi}_t$ from the statistic

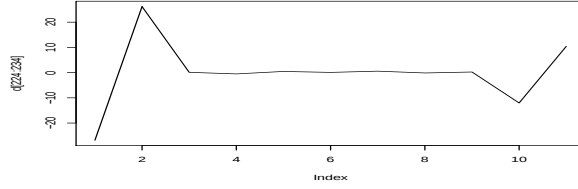
$$\Delta\hat{\xi}_t = \Delta \frac{\widehat{icov_{t,1}}(u, v)}{y_t^2}, \quad y_t \neq 0$$

is computed and displayed graphically to detect periods when it is approximately constant and close to 0. The statistic $\Delta\hat{\xi}_t$ indicates the times when the process becomes a "tail process", which can be used to approximate the start and end of a bubble. The bottom panel shows the first difference $\Delta\hat{\xi}_t$. The first difference $\Delta\hat{\xi}_t$ is constant and close to zero during the entire bubble episode.

Figure 2: Bubble detection in MAR(1,1) process



(a) MAR(1,1) process



(b) Path of $\Delta\xi_t$

Figure 3 below shows an example of the path of a simulated MAR(1,1) process with $\phi = 0.3$ and $\psi = 0.9$, and i.i.d. errors from a $t(3)$ distribution.

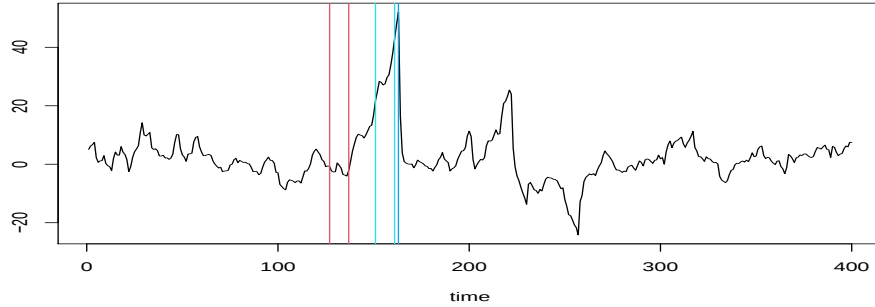


Figure 3: Simulated Path

The vertical lines mark times 127, 137 (red), 151, 161 (blue), 163(green)

We observe a bubble with a peak of 52.051 at $t = 163$. The estimates of the GCov parameter of the MAR(1,1) model are $\hat{\phi}=0.3085$ and $\hat{\psi}=0.908$, with standard errors of 0.028 and 0.030, and these estimates are based on $K = 4$ power transformations and $H = 3$. The test statistic $\hat{\xi}_{t,T}$ evaluated during the bubble, conditional on observation $y(160) = 37.951$ is -0.0220 and is within the confidence interval of ± 0.05646 . Therefore, the null hypothesis $\theta_1 = 0$ at time $t = 160$ is not rejected.

Next, we examine the effect of the conditioning values and perform the test using $\hat{\xi}_{t,T}(h)$ on increasing horizons $h=1$ to 10, conditional on $y_{127} = -0.7522$ and $y_{151} = 21.810$. The results

are reported in Table 2. The columns of Table 1 report: the horizon (col. 1), the value of the test statistic (cols. 2 and 5), the confidence interval of the form $0.0 \pm CI$, and the result of the test of $H_{0,h} : \theta_h = 0$ coded 1 for not rejected and 0 for rejected. We observe that, conditional on the large value at time $t = 151$, the test does not reject the null hypothesis of a bubble over the upward-sloping sequence of the next 10 observations. Conditional on a close proximity to the mean value of the process at time $t = 127$, the test rejects the null hypothesis of a bubble over the next 10 observations, which remain close to the mean.

To evaluate the performance of the test in finite samples, we perform the following experiment. We generate MAR(0,1) processes in samples of length $T=400$ ⁵ with $t(3)$, $t(4)$, and $t(5)$ distributed errors and with different values of coefficients ϕ and ψ .

Table 1: Test at horizons 1 to 10

horizon	Conditional on y_{127}			Conditional on y_{151}		
h	$\hat{\xi}_{t,T}(h)$	CI	test	$\hat{\xi}_{t,T}(h)$	CI	test
1	-4.7826	± 0.5263	0	-0.0322	± 0.0581	1
2	-4.5177	± 0.5720	0	-0.0365	± 0.0746	1
3	-4.1350	± 0.5166	0	-0.0340	± 0.0686	1
4	2.9267	± 0.1242	0	-0.0326	± 0.0637	1
5	-1.1938	± 0.1028	0	-0.0338	± 0.0671	1
6	-2.7478	± 0.2047	0	-0.0374	± 0.0799	1
7	-7.2927	± 1.1972	0	-0.0377	± 0.0829	1
8	-6.1137	± 1.0591	0	-0.0432	± 0.1054	1
9	-6.7780	± 1.2766	0	-0.0485	± 0.1324	1
10	-2.0080	± 0.2329	0	-0.0552	± 0.1704	1

Next, we set $y = q_y(0.975)$ to be used as a conditioning large positive threshold value at time t to study the size of the test. To study the power of the test, we use $y = q_y(0.525)$ as the conditioning value at time t followed by a moderate decrease or increase, and preceded by a random pattern, where by moderate we mean that y_{t+1} is less than or equal to $q_y(0.975)$. The empirical size and power of the test are reported in Tables 4 and 5, Appendix C. Table 6 considers the MAR(1,1) with Cauchy distributed errors. A similar exercise is performed for the MAR(1,1) processes with t-student distributed errors, estimated by the GCov with $K = 2$ power transformations and lag $H = 4$. The size and power are reported in Tables 6 and 7, Appendix C.

There are some challenges involved in this analysis. It is difficult to accurately estimate the quantiles of the marginal distribution of the process from 400 observations, especially for processes with high persistence and low degrees of freedom. Hence, the estimates of

⁵We generate a series of length 800 and discard the first and last 200 observations.

$y = q_y(0.975)$ as the conditioning value, obtained from the replicated paths of the process, may vary. Moreover, without observing the trajectory, we do not know a priori whether the process admits a bubble or a jump instead, the latter occurring in the MAR(1,1) processes due to the causal component. In processes with $\psi = 0.9$, bubbles are infrequent and grow slowly at a rate of about 1.1⁶. In processes with $\psi = 0.6$, the bubbles are more frequent and grow faster. We distinguish the bubble from jumps by checking if, following the conditioning value of $y = q_y(0.975)$, the process grows at a rate close to the theoretical rate. Another difficulty is the asymptotic normality of the estimators, which impacts the results in the processes with $t(3)$ distributed errors estimated by the OLS. In Table 4, we observe that the test over-rejects when the error distribution is $t(5)$, especially for higher values of ψ . Table 5 shows that the power of the test is slightly lower when the noncausal persistence is higher.

Table 6, illustrating processes with $\psi = 0.9$ and Cauchy distributed error, shows that the test tends to under-reject for closer to 0.9 values of parameters ϕ . In addition, the power of the test decreases for higher values of ϕ .

In Table 7, we observe that the size of the GCov-based test is closer to the nominal value for moderate values of $\psi = 0.7$ and $\psi = 0.8$. The test is conservative for $\psi = 0.6$ and over-rejects for $\psi = 0.9$ in processes with $t(3)$ distributed errors. For each ψ , the test is the most conservative in processes with $t(5)$ distributed errors. This error distribution is the closest to the normal among the three error distributions considered, leading to potential parameter identification problems. We find that the size is close to the nominal value for moderate values of causal persistence ϕ , regardless of ψ . Table 8 shows that the test has good power, especially for $\psi = 0.7$ and $\psi = 0.8$. It deteriorates in processes with $t(5)$ distributed errors for $\psi = 0.6$ and $\psi = 0.9$ and for higher values of causal persistence, making bubbles harder to distinguish from regular dynamics.

5 Green Stock Indexes and ETFs

The transition to a clean energy economy has recently gained significant attention, driven by global events such as the COVID-19 pandemic and the Russia-Ukraine war. In fact, in response to these crises, governments, industries, and investors have increasingly prioritized clean energy investments, recognizing their long-term benefits for a stable and environmentally friendly energy system [Mohammed et al. (2023)]. This growing focus on clean energy is also driven by the need to achieve net zero emissions by 2050, which requires substantial

⁶The tests of Phillips et al. (2011) and Phillips et al. (2015) are intended to test bubbles with a growth rate of 1. Gourioux and Zakoian (2017) show that unit root tests are unreliable in applications to a MAR(0,1) process with Cauchy-distributed errors.

investment in clean energy from both developed and developing countries; see [Khalifa et al. \(2022\)](#). In 2023, global investment in the energy sector was estimated at USD 2.8 trillion, an increase of 0.6 trillion USD from five years earlier. Almost all of this growth was directed toward clean energy and infrastructure, increasing total clean energy spending to 1.8 trillion USD, compared to 1 trillion USD for fossil fuels.⁷ These large investments pose the risk of "green bubbles" – rapid stock price increases followed by crashes. Specifically, such bubbles occur when overinvestment and speculative behavior drive the market value of clean energy assets beyond sustainable levels. Consequently, rising interest rates can increase financial pressure on investors, potentially triggering bubble bursts and undermining the credibility of the clean energy transition [[Wimmer \(2016\)](#)].

5.1 The Data

We have considered the Renixx Index, the WHETF, and the iShare green stock ETFs. For Renixx, we used daily data from <https://www.renewable-energy-industry.com/stocks> and sampled them at a monthly frequency by taking the last day of each month of the closing price series. When the last day of the month is a bank holiday, we consider the previous available day. The Renixx index tracks the global renewable energy market, covering sectors such as wind, solar, bioenergy, geothermal, hydropower, electronic mobility, and fuel cells. It comprises 30 companies, each of which derives more than 50% of its revenues from these sectors (see www.iwr.de/renixx). The WHETF tracks the WilderHill Clean Energy ETF, which includes companies listed in the United States that focus on developing cleaner energy and conservation efforts. In particular, WHETF allocates a minimum of 90% of its total assets to common stocks included in this ETF. It is rebalanced and reconstituted quarterly. Finally, iShare tracks the S&P Global Clean Energy ETF, which provides exposure to the 30 largest and most liquid publicly traded companies worldwide operating in the clean energy sector. The latter uses a modified market capitalization weighting system.

We investigate Renixx on its whole available sample and both WHETF and iShare from 2009 onward. We consequently have a total of $T = 266$ observations for Renixx and $T = 182$ observations for WHETF and iShare, with potentially two bubble patterns for Renixx and a single bubble for both the WHETF and iShare. Figure 4a shows that the Renixx index (from January 2002 to February 2024) experienced significant bubbles in 2008 and 2020, coinciding with two major global events: the 2008 financial crisis and the outbreak of the COVID-19 pandemic. In 2008, the financial crisis rocked global markets, leading to widespread economic

⁷IEA (2023), World Energy Outlook 2023, IEA, Paris <https://www.iea.org/reports/world-energy-outlook-2023>, License: CC BY 4.0 (report); CC BY NC SA 4.0 (Annex A).

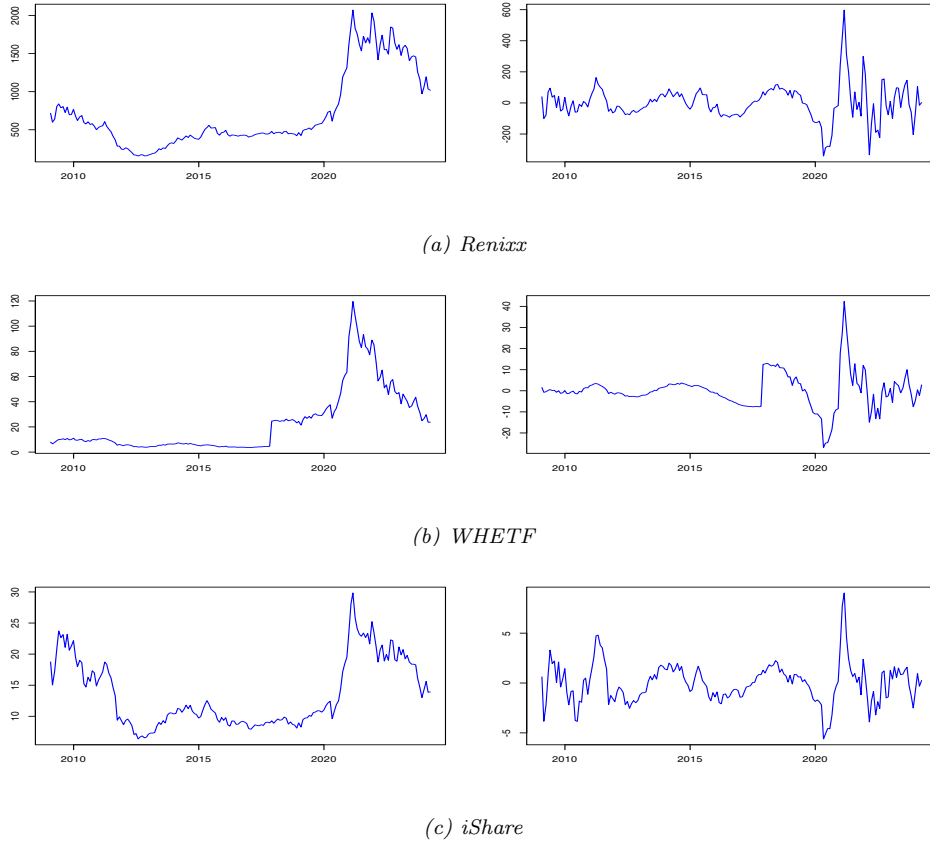
instability and investor panic. The collapse of major financial institutions, coupled with a credit crunch and falling stock markets, triggered a flight to safety among investors. This risk aversion had a significant impact on the renewable energy sector, with reduced investment in renewable energy projects and a decrease in the demand for renewable energy stocks [see [Giorgis et al. \(2024\)](#)]. Similarly, in 2020, the COVID-19 pandemic caused unprecedented economic disruption around the world. Lockdown measures, supply chain disruptions, and reduced consumer spending resulted in a global economic downturn. The renewable energy sector was particularly affected by the decrease in energy demand due to reduced economic activity and travel restrictions. The same trends are observed in WETHF and iShare; Figures 4b and 3c, respectively. However, only the COVID period bubble appears in the data, as the sample for both variables spans from January 2009 to February 2024, and therefore does not include data from the global financial crisis.

5.2 Prices vs. Returns

Although most studies in green energy finance focus on returns [e.g., [Henriques and Sadorsky \(2008\)](#); [Sadorsky \(2012a\)](#)], our analysis is based on prices, allowing for a direct examination of bubble dynamics that would otherwise be obscured by log-differencing. In fact, in financial analysis, especially in commodity pricing, it is crucial to focus on the price process rather than returns or first differences. This approach aligns with the financial theory that underlies commodity pricing, as discussed in [Hull and Basu \(2016\)](#). Transforming the price series into returns or first differences can obscure critical aspects of the price process, including the presence of bubbles. Moreover, differencing can eliminate noncausal components essential to understanding bubble dynamics [[Giancaterini et al. \(2022\)](#)]. We employ the detrended cubic spline method to address this issue and eliminate trend components while preserving significant bubble patterns, as detailed in [Hall and Jasiak \(2024\)](#). This method fits a cubic spline, a piecewise function composed of polynomial segments of degree three, to the time series data. The points where these segments connect, known as knots, allow separate cubic polynomials to be fitted within each segment. We place knots every two years to effectively detrend the series, balancing data smoothing and avoiding overfitting. By applying this approach, we successfully isolate the cyclical variations that contain the bubble patterns from the trend component. The detrended series are also shown in the right panels of Figure 4.⁸

⁸An alternative detrending technique that may help us preserve the bubble patterns is the HP filter method (see [Giancaterini et al. \(2022\)](#), [Hecq and Voisin \(2023\)](#)). However, as indicated by [Hall and Jasiak \(2024\)](#), the HP filter requires you to choose a value for the smooth parameter lambda, which is typically a

Figure 4: Green Index and ETFs from January 2009 to February 2024. The panels on the left show the original series, while those on the right display their detrended versions.



5.3 Summary Statistics and MAR Estimations

The detrended data is non-Gaussian, as evidenced by the Kolmogorov-Smirnov and Shapiro-Wilk's tests, which both reject the null hypothesis of normality. Table 2 presents the summary statistics, which confirm that the distributions of the series are non-Gaussian, given the reported excess kurtosis. We test the spline-detrended data for causal and noncausal persistence by using the test introduced in Jasiak and Neyazi (2023) [see Section 2.2]. We choose $K = 2$, including the time series and its squares as (non)linear transformations, and $H = 3$ as the number of lags in the objective function of the test. The null hypothesis of the absence of nonlinear serial dependence is rejected since the test value is 514.98, while the

function that increases with the sampling frequency of the data. In our examination of monthly data, a high lambda value may result in computational inaccuracies, thus justifying our preference for using a cubic spline to detrend the data. However, we have investigated several detrending methods, including the HP filter, as well as polynomial trends of different degrees (results upon request). We only report the results with the spline detrending approach as it passes the absence of nonlinear serial dependence test [Jasiak and Neyazi (2023)].

critical value at a 5% significance level from the chi-square distribution is $\chi^2(12) = 21.026$.

Table 2: Summary statistics of Renixx, WHETF, and iShare

	T	Mean	Min	Max	SD	Skewness	Ex.Kurt.
Renixx	266	748.57	155.47	2070.38	475.09	1.03	2.85
WHETF	182	23.08	3.66	119.57	25.31	1.67	5.28
iShare	182	13.60	6.36	29.83	5.46	0.74	2.36

We apply the GCov estimator to obtain the parameters of MAR models for the three series. Specifically, we use $H = K = 2$ and $a_j(\epsilon_t) = \epsilon_t^j$, for $j = 1, 2$, in (5), i.e., the residuals and their squares as transformations in the GCov estimation of Renixx [see Cubadda et al. (2023)] and $a_j(\epsilon_t) = \log(|ps|)^j$ for $j = 1, 2$, for WHETF and iShare. Table 3 presents the estimation results. To identify the dynamics of the underlying processes, we evaluate all combinations of r and s such that $r + s = p$. We begin with $p = 1$ and gradually increase p to find the values of r and s that provide i.i.d. residuals based on the GCov specification test. The last row of 3 shows the results of the GCov test. This approach allows us to identify the Renixx index and the two ETFs as MAR(1, 1) processes. The GCov specification test results indicate that these processes are correctly specified and provide a good fit to the data.

Figure 5: Causal and noncausal components

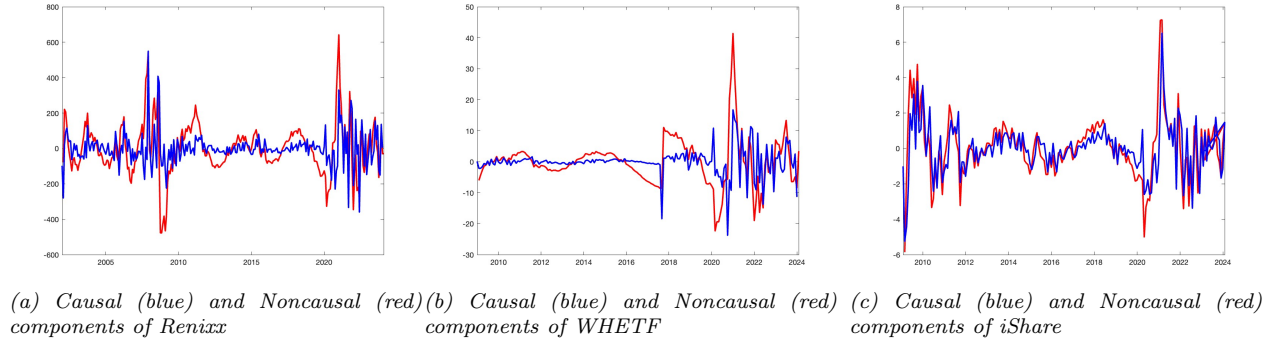


Figure 5 shows the estimated latent components $\hat{u}_t$ and $\hat{v}_t$ defined in eq. (3) and (4) for the three series of interest. We observe that the changes in the noncausal component anticipate the dynamics of the observed series, followed by the causal component. In the next section, these estimates are used to compute the statistic $\xi_{t,T}$ for each series.

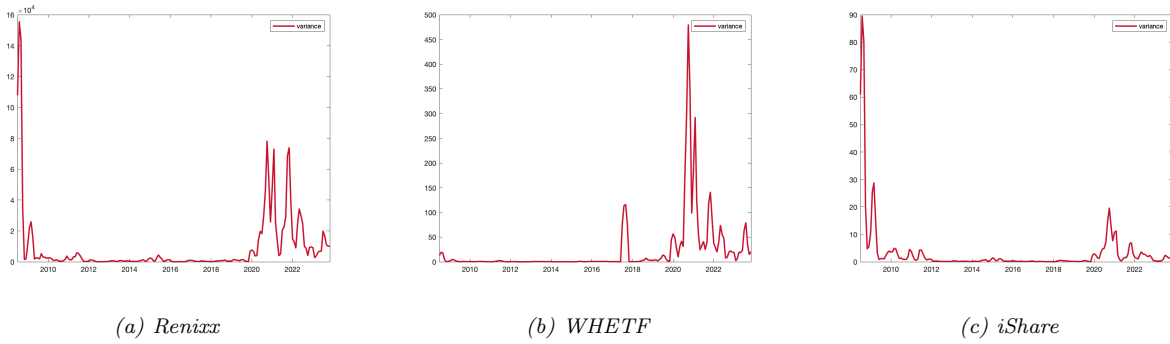
Table 3: Estimated $MAR(1,1)$ coefficients and $GCov$ tests

$MAR(1,1)$	Renixx	WHETF	iShare
ϕ	0.24 (0.11)	0.07 (0.01)	0.32 (0.02)
ψ	0.70 (0.08)	0.89 (0.03)	0.62 (0.02)
GCov Test (Critical Value)	7.14 (12.59)	4.86 (12.59)	2.40 (12.59)

5.4 Bubble Detection

The bubbles in the price series can manifest themselves as periods of high volatility. We estimate local variances to detect periods of high volatility in Renixx and the ETFs. We consider a rolling window of 5 observations and estimate the local variance for the three series. The plot of rolling estimates provides a graphical tool for preliminary analysis. Figure 6 displays the rolling estimates of the variances of the Renixx index and two ETFs. Two bubble periods, one that occurred during the financial crisis of 2008 and one associated with the COVID pandemic, are observable in Figure 6. Since the data on ETFs before 2009 are not available and the observations on bubbles during that period are incomplete, we focus on the bubble observed during the COVID period.

Figure 6: Local variance of time series with $H = 5$ rolling window size

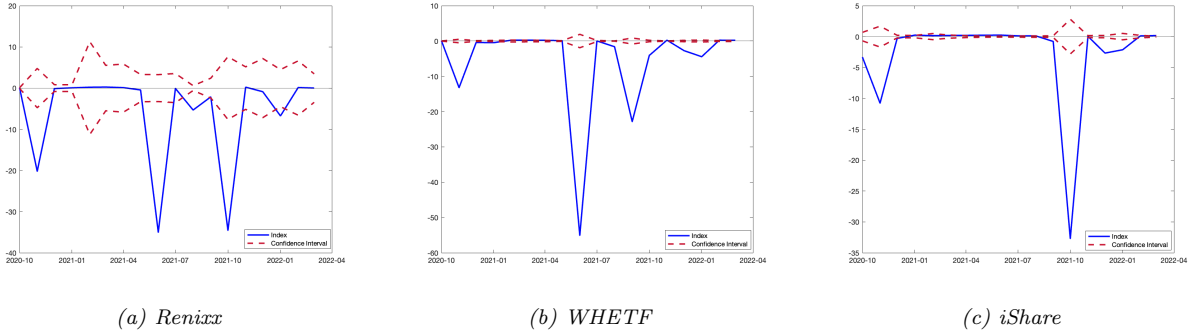


To detect and determine the dates of bubbles in the three price series, we focus specifically on the period between 12/01/2020 and 05/31/2021. Figure 7 illustrates how the method introduced in Section 4 can be applied to Renixx and the ETFs to test for bubbles, and to determine the dates at which that bubble starts and ends. Figure 7a and 7b show the test statistics $\hat{\xi}_{t,T}$ computed over that period and the associated confidence band. We observe

that the bubble of Renixx and WTETF started on 12/01/2020 and ended on 03/01/2021. The bubble for ishare ended on 04/30/2021. Note that in Figure 7c the confidence intervals depend on the parameter variance reported in Table 3 and are larger for Renixx compared to the other estimated processes.

Given a tail index value, the time to peak depends on the rate of bubble growth. It takes longer for the bubble to grow when $\hat{\psi}$ is large. As shown in Section 3.3, the time to peak is also determined by the tail parameter, which is estimated from the Hill estimator R-package. For $\hat{\alpha} = 1.3$, the expected time to peak for Renixx is 1.5 months (computed as the conditional expectation of N given $N < 0$), and the probability that N exceeds 3 months is 0.232.

Figure 7: Bubble detection statistics for Renixx, WHETF, and ishare



Given $\hat{\alpha} = 1.7$ for WHETF, the expected time to peak is 4.5 months, and the probability that it exceeds 5 months is 0.37. With $\hat{\alpha} = 1.15$, the time to peak for iShare is slightly longer than 1 month, and the probability that it exceeds 3 months is 0.17.

6 Conclusions

This paper introduced a new statistical test for detecting financial bubbles, derived from the tail process representation of mixed causal–noncausal autoregressive models. Unlike traditional unit-root and explosiveness-based methods, our approach is grounded in strictly stationary noncausal dynamics, which provide a natural framework for capturing locally explosive behavior. Classical tests such as SADF or GSADF are often sensitive to heavy tails or may misinterpret short-lived spikes as bubbles. By contrast, our test exploits the tail process properties of causal and noncausal dynamics, which remain constant and close to zero during a bubble episode. This feature allows us to separate persistent speculative bubbles from short-lived spikes and regular fluctuations.

We applied univariate $\text{MAR}(r, s)$ models to the Renixx index and two green energy ETFs

(WHETF and iShare), estimating the underlying dynamics using the GCov approach. This combination provides a robust identification strategy: the GCov estimator ensures reliable parameter estimation under non-Gaussianity, while the proposed test captures the tail process behavior that signals bubble formation. The empirical results reveal clear bubble patterns in all three series, showing that the test performs well in detecting and dating speculative episodes in green financial markets.

Appendix A

1. Derivation of the MA(∞) representation of MAR(1,1)

We have:

$$(1 - \phi L)(1 - \psi L^{-1}) = -\psi L^{-1}(1 - \phi L)(1 - \psi^{-1}L).$$

It follows using the partial fraction decomposition of the polynomial that:

$$\begin{aligned} \frac{1}{(1 - \phi L)(1 - \psi L^{-1})} &= -\psi^{-1}L \frac{1}{(1 - \phi L)(1 - \psi^{-1}L)} \\ &= \frac{-\psi^{-1}L}{\phi - \psi^{-1}} \left(\frac{\phi}{1 - \phi L} - \frac{\psi^{-1}}{1 - \psi^{-1}L} \right) \\ &= \frac{1}{1 - \phi\psi} \left(\frac{\phi L}{1 - \phi L} - \frac{\psi^{-1}L}{1 - \psi^{-1}L} \right) \\ &= \frac{1}{1 - \phi\psi} \left(\frac{\phi L}{1 - \phi L} + \frac{1}{1 - \psi L^{-1}} \right) \\ &= \frac{1}{1 - \phi\psi} \left(\sum_{h=1}^{\infty} \phi^h L^h + \sum_{h=0}^{\infty} \psi^h L^{-h} \right) \\ &= \frac{1}{1 - \phi\psi} \left(\sum_{h=1}^{\infty} \phi^h L^h + \sum_{h=0}^{-\infty} \psi^{-h} L^h \right), \end{aligned}$$

where we observe that the first sum inside the brackets is equal to $\frac{1}{1 - \phi L} - 1$, and the second sum is equal to $\frac{1}{1 - \psi L^{-1}}$. The moving average coefficients are:

$$c_h = \frac{\phi^h}{1 - \phi\psi}, \text{ if } h > 0, \quad \text{and} \quad c_h = \frac{\psi^{-h}}{1 - \phi\psi}, \text{ if } h \leq 0,$$

since the two formulas coincide for $h = 0$ and the convention $0^0 = 1$ is used (if ϕ or ψ is zero).

2. Proof of Proposition 2:

The tail process takes the values either $1/\psi X_{h-1}$, or ϕX_{h-1} . Then, we have:

$$\text{a) if } N + h \leq 0 \iff N \leq -h : X_h = \frac{1}{1 - \phi\psi} \psi^{-N-h} / \frac{1}{1 - \phi\psi} \psi^{-N-h-1} X_{h-1} = \psi^{-1} X_{h-1},$$

$$\text{b) if } N + h > 0 \iff N > -h : X_h = \frac{1}{1 - \phi\psi} \phi^{N+h} / \frac{1}{1 - \phi\psi} \phi^{N+h-1} X_{h-1} = \phi X_{h-1},$$

with $X_0 = 1$.

Then, the probability distribution of N is such that :

$$P[N = h] = \frac{(1 - \phi\psi)^{-1} \psi^{-h\alpha}}{(1 - \phi\psi)^{-1} [\frac{1}{1 - \phi^\alpha} + \frac{1}{1 - \psi^\alpha} - 1]} = \frac{\psi^{-h\alpha}}{[\frac{1}{1 - \phi^\alpha} + \frac{1}{1 - \psi^\alpha} - 1]}; \text{ if } h \leq 0,$$

$$P[N = h] = \frac{(1 - \phi\psi)^{-1} \phi^{h\alpha}}{(1 - \phi\psi)^{-1} [\frac{1}{1 - \phi^\alpha} + \frac{1}{1 - \psi^\alpha} - 1]} = \frac{\phi^{h\alpha}}{[\frac{1}{1 - \phi^\alpha} + \frac{1}{1 - \psi^\alpha} - 1]}; \text{ if } h \geq 0,$$

with $0^0 = 1$, by convention.

3. Tail process representation of AR(2) processes with at least one root inside the unit circle

The autoregressive of order 2, AR(2), process:

$$y_t = t_1 y_{t-1} + t_2 y_{t-2} + \epsilon_t \Rightarrow y_t(1 - t_1 L - t_2 L^2) = \epsilon_t$$

with i.i.d. errors $\epsilon_t, t = 1, 2, \dots$ from a heavy-tailed distribution with tail index α can admit a tail process behavior. We distinguish the following cases:

a) $1 - t_1 L - t_2 L^2 = (1 - \lambda_1 L)(1 - \lambda_2 L)$, the root reciprocals λ_1 and λ_2 are real, distinct, and such that λ_1 and λ_2 are real, distinct, and such that $|\lambda_1| < 1$ and $|\lambda_2| < 1$ (causal process), or $|\lambda_1| > 1$ and $|\lambda_2| > 1$ (noncausal process). Then, the process can be written as a strictly stationary pure causal or noncausal process.

The causal process admits a one-sided MA(∞) representation with the coefficients $c_h = \frac{1}{\lambda_1 + \lambda_2}[(\lambda_2)^h + (\lambda_1)^h]$, for $h \geq 0$. The noncausal process admits a one-sided MA(∞) representation with coefficients $c_h = \frac{1}{\lambda_1 - \lambda_2}[(\lambda_2)^{h+1} - (\lambda_1)^{h+1}]$, for $h \leq 0$

b) $1 - t_1 L - t_2 L^2 = (1 - \lambda L)^2$, the root reciprocal λ is real and such that either $|\lambda| < 1$ (double causal root), or $|\lambda| > 1$ (double noncausal root). This case does not satisfy the geometric ergodicity condition and is excluded.

d) $1 - t_1 L - t_2 L^2 = (1 - \lambda_1 L)(a - \lambda_2 L)$, the roots $1/\lambda_1$ and $1/\lambda_2$ are imaginary and equal to a pair of complex conjugates of modulus greater than 1. This case is excluded as it results in local explosive oscillations, while a bubble requires local explosive growth.

4. Comparison with Fries (2022) For illustration, let us consider $h = 0$ and the MAR(1,1) process with α -stable distributed errors using the approach of Fries (2022). It follows from Propositions 1 and 2 that, conditional on a bubble onset at time t and large $y_t > y$ we have:

$$\begin{aligned} E\left(\frac{u_{t+1}v_t}{y_t^2} | y_t > y\right) &= E\left(\left(\frac{y_{t+1} - \phi y_t}{y_t}\right)\left(\frac{y_t - \psi y_{t+1}}{y_t}\right) | y_t > y\right) \\ &\approx E((X_1 - \phi)(1 - \psi X_1)) \\ &= p^+ (\psi^{-1} - \phi) (1 - \psi \psi^{-1}) + p^- (\phi - \phi) (1 - \psi \phi) \\ &= 0 \end{aligned}$$

where we replace $\frac{y_t}{y_t}$ by $X_0 = 1$, $\frac{y_{t+1}}{y_t}$ by X_1 , independent of t and y , and the probabilities p^+ and p^- for $h \leq -1$ and $h \geq 0$, of bubble growth and burst, respectively, are defined in Proposition 2.

Let us consider $h = 0$ and the MAR(0,1) process. In this process, component $u_t = 1$ and only the component v_t is relevant. Hence, conditional on the bubble onset at time t for large $y_t > y$ we have:

$$\begin{aligned} E\left(\frac{u_{t+1}v_t}{y_t} | y_t > y\right) &= E\left(\left(\frac{y_t - \psi y_{t+1}}{y_t}\right) | y_t > y\right) \\ &\approx E(1 - \psi X_1) \\ &= p^+ (1 - \psi \psi^{-1}) \\ &= 0, \end{aligned}$$

where we replace $\frac{y_{t+1}}{y_t}$ by X_1 , independent of t and y , $\frac{y_t}{y_t} = X_0 = 1$ and the probability p^+ of bubble growth is given in Corollary 2.

5. The cumulative probability distribution of N

a) For the MAR(1,1) process:

If $h < 0$, we have:

$$\begin{aligned} P[N \leq h] &= \sum_{i \leq h} \frac{\psi^{-i\alpha}}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]} \\ &= \sum_{i \geq -h} \frac{\psi^{i\alpha}}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]} \\ &= \frac{\psi^{-h\alpha}}{1 - \psi^\alpha} \frac{1}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]} \end{aligned}$$

If $h > 0$, then:

$$P[N > h] = \frac{\phi^{h\alpha}}{1 - \phi^\alpha} \frac{1}{\left[\frac{1}{1-\phi^\alpha} + \frac{1}{1-\psi^\alpha} - 1\right]}$$

b) For the MAR(0,1) process with $N \leq 0$ and $h \leq 0$, we have:

$$P[N \leq h] = (1 - \psi^\alpha) \sum_{i \leq h} \psi^{-i\alpha} = (1 - \psi^\alpha) \sum_{i \geq -h} \psi^{i\alpha} = \psi^{-h\alpha}$$

.

Appendix B

This Appendix describes the conditional covariance of the latent components when the process does not take extreme values. For ease of exposition, consider the MAR(1,1) process defined in Section 2.1.

The latent components u_t, v_t are strictly stationary. When ϵ_t has finite moments of orders 1 and 2, the latent components have marginal means $E(u_t) = E(v_t) = 0$ and their marginal covariance is always known to be zero $E(u_{t+1}v_t) = 0$.

Let us consider the conditional moments that exist regardless of the existence of marginal moments. Since the MAR(1,1) process is Markov of order 2 [see, e.g. [Fries and Zakoian \(2019\)](#), Proposition 3.1], the conditioning set considered is $\underline{y}_t = \{y_t, y_{t-1}\}$. Then $E(u_{t+1}|\underline{y}_t) = E[y_{t+1}|\underline{y}_t] - \phi y_t$ and $E(v_t|\underline{y}_t) = y_t - \psi E[y_{t+1}|\underline{y}_t]$. To interpret the statistic ξ , we write the conditional covariance of latent components $E(u_{t+1}v_t|\underline{y}_t)$:

$$E(u_{t+1}v_t|\underline{y}_t) = E[(y_{t+1} - \phi y_t)(y_t - \psi y_{t+1})|\underline{y}_t] = -\phi y_t^2 - \psi E[y_{t+1}^2|\underline{y}_t] + (\phi\psi + 1)y_t E[y_{t+1}|\underline{y}_t].$$

and observe that it depends on the autoregressive coefficients ϕ and ψ , and the squared values of the process. [Gourieroux and Zakoian \(2017\)](#) and [Fries and Zakoian \(2019\)](#) show that MAR(r,s) processes display conditional heteroskedasticity. More specifically, the conditional volatility of a MAR(1,1) process with Cauchy distributed errors is a quadratic function of the past values of the process, similar to the conditional heteroscedasticity of the Double Autoregressive (DAR) model of [Ling \(2004\)](#) with variance-induced mean reversion. The conditional covariance of the latent components given above depends on the squared past values of the process, and therefore captures the conditional heteroscedasticity of the MAR process, in addition to its conditional autocovariance.

For the MAR (1,1) process with Cauchy-distributed errors and $\psi > 0$, the formulas of conditional covariances are easy to compute. We know the equivalence of the conditioning sets $\underline{y}_t = \underline{u}_t$ and from Proposition 5, [Gourieroux, Zakoian](#) [Gourieroux and Zakoian \(2017\)](#), it follows that:

$$E(u_{t+1}|\underline{u}_t) = u_t, \quad E(u_{t+1}^2|\underline{u}_t) = \frac{1}{\psi} u_t^2 + \frac{\sigma^2}{\psi(1-\psi)}.$$

Since $u_{t+1} = y_{t+1} - \phi y_t$, we can write $y_{t+1} = \phi y_t + u_{t+1}$. Hence,

$$E(y_{t+1}|\underline{y}_t) = E(u_{t+1}|\underline{y}_t) + \phi y_t = u_t + \phi y_t = (y_t - \phi y_{t-1}) + \phi y_t$$

and from Proposition M.1, [Fries and Zakoian \(2019\)](#) we get:

$$E[y_{t+1}^2 | \underline{y}_t] = ay_t^2 - 2by_t y_{t-1} + cy_{t-1}^2 + \frac{\sigma^2}{|\psi|(1-|\psi|)}$$

where $a = \phi^2 + 2\phi \operatorname{sign}(\psi) + \frac{1}{|\psi|}$, $b = \phi^2 \operatorname{sign}(\psi) + \frac{\phi}{|\psi|}$ and $c = \frac{\phi^2}{|\psi|}$. For clarity of exposition, let us assume $\psi > 0$. Then, we get:

$$\begin{aligned} E[(u_{t+1}v_t) | \underline{y}_t] &= -\phi y_t^2 - \psi \left(ay_t^2 - 2by_t y_{t-1} + cy_{t-1}^2 + \frac{\sigma^2}{|\psi|(1-|\psi|)} \right) + (\phi\psi + 1)(y_t^2 + (1-L)\phi y_t^2) \\ &= y_t^2[(\phi - 1)\phi\psi] + y_t y_{t-1}[2\phi - \phi(1 - \phi\psi)] - y_{t-1}^2 + \frac{\sigma^2}{(1-|\psi|)} \end{aligned}$$

The conditional covariance of the latent components is equal to the constant $\frac{\sigma^2}{(1-|\psi|)}$ when $y_t = y_{t-1} = 0$ and it is time-varying otherwise. In particular, for $E[(u_{t+1}v_t) | \underline{y}_t]$ the ratio

$$E_t \xi_t = \frac{E[(u_{t+1}v_t) | \underline{y}_t]}{y_t^2} \text{ for } y_t \neq 0$$

is time-varying. In general, for $h > 1$ we have $E(u_{t+1+h}v_{t+h} | \underline{y}_t)$ where

$$\begin{aligned} E(u_{t+1+h}v_{t+h} | \underline{y}_t) &= E[(y_{t+1+h} - \phi y_{t+h})(y_{t+h} - \psi y_{t+1+h}) | \underline{y}_t] \\ &= -\phi E[y_{t+h}^2 | \underline{y}_t] - \psi E[y_{t+h+1}^2 | \underline{y}_t] + (\phi\psi + 1)E[y_{t+h}y_{t+h+1} | \underline{y}_t] \end{aligned}$$

The conditional covariance of the latent components is a function of the conditional second moments and the conditional covariance of y_t at lag h . The first two terms of $\operatorname{cov}_t(u, v)$ capture the time-varying conditional heteroscedasticity of y_t . For positive ϕ, ψ these terms become big and negative when the conditioning value is large. The third term is the conditional autocovariance at horizon h . The conditional second moments and conditional autocovariances at lag h can be found from the formula of $E[y_t y_{t+h} | \underline{y}_t]$ given in Proposition M.1, [Fries and Zakoian \(2019\)](#) and obtained from the following representation of the process:

$$y_{t+h} = P_h(L)y_{t-1} + Q_h(L^{-1})u_t,$$

where for $h = 0$, we have $y_t = \phi y_{t-1} + u_t$ with $P_0(L) = \phi$ and $Q_0(L^{-1}) = 1$. For $h = 1$, we have

$$y_{t+1} = \phi^2 y_{t-1} + \phi u_t + u_{t+1}$$

with $P_1(L) = \phi^2$ and $Q_1(L^{-1}) = \phi + L^{-1}$. Next, $y_{t+2} = \phi^3 y_{t-1} + u_{t+2} + \phi u_{t+1} + \phi^2 u_t$, with $P_2(L) = \phi^3$ and $Q_2(L^{-1}) = L^{-2} + L^{-1} + \phi^2$, etc.

A Appendix C

Table 4: Empirical size of the test for $MAR(0,1)$ estimated by OLS (1000 replications) at 5% from samples of $T=400$. Each row corresponds to an error distribution: $t(3)$, $t(4)$, $t(5)$. Columns give the rejection frequency (Size) for different values of the noncausal autoregressive coefficient ψ (from 0.1 to 0.9). The statistic is evaluated conditional on extreme events, defined as observations exceeding the 97.5th percentile of the simulated series.

Distribution	ψ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.012	.012	.012	.014	.015	.013	.035	.053	.076
$t(4)$	.050	.021	.029	.027	.045	.051	.069	.088	.122
$t(5)$	.154	.102	.056	.072	.091	.085	.115	.133	.138

Table 5: Empirical power of the test for $MAR(0,1)$ estimated by OLS (1000 replications) at 5% from samples of $T=400$. Each row corresponds to an error distribution: $t(3)$, $t(4)$, $t(5)$. Columns give the rejection frequency (Power) for different values of the noncausal autoregressive coefficient ψ (from 0.1 to 0.9). The statistic is evaluated at the observation corresponding to the 210th largest value among 400 simulated observations (52.5th percentile).

Distribution	ψ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.592	.582	.555	.515	.499	.531	.486	.511	.496
$t(4)$	.579	.589	.574	.544	.541	.537	.523	.502	.512
$t(5)$	.561	.585	.559	.525	.517	.512	.500	.514	.472

Table 6: Empirical size and power of the test for $MAR(1,1)$ with Cauchy distributed errors estimated by GCov (1000 replications) at 5% from samples of $T=400$. Columns give the rejection frequency (Size) for different values of the noncausal autoregressive coefficient ψ (from 0.1 to 0.9). The statistic is evaluated conditional on extreme events, defined as observations exceeding the 97.5th percentile of the simulated series (size), and 210th largest value among 400 simulated observations (52.5th percentile) (power).

$\psi = 0.9$									
ϕ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
size	0.08	0.05	0.04	0.08	0.04	0.06	0.04	0.01	0.01
power	0.72	0.84	0.86	0.68	0.56	0.62	0.54	0.46	0.47

Table 7: Empirical size of the test for $MAR(1,1)$ estimated by $GCov$ (1000 replications) at 5% from samples of $T=400$. Rows correspond to an error distribution: $t(3)$, $t(4)$, $t(5)$. Columns report the rejection frequency (Size) for different values of the causal coefficient ϕ (0.1–0.9) with ψ fixed. The statistic is evaluated conditional on extreme events, defined as observations exceeding the 97.5th percentile of the simulated series.

$\psi = 0.6$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.015	.015	.012	.010	.010	.012	.015	.016	.012
$t(4)$	.006	.007	.008	.006	.006	.008	.011	.010	.005
$t(5)$	.006	.004	.002	.002	.002	.002	.004	.004	.002
$\psi = 0.7$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.049	.064	.055	.057	.049	.041	.045	.050	.051
$t(4)$	.023	.025	.033	.031	.028	.023	.031	.040	.036
$t(5)$	.016	.021	.018	.016	.013	.012	.012	.022	.034
$\psi = 0.8$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.056	.052	.053	.052	.054	.049	.046	.043	.077
$t(4)$	.036	.033	.034	.036	.040	.044	.043	.038	.065
$t(5)$	.025	.020	.028	.028	.029	.027	.027	.023	.052
$\psi = 0.9$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.055	.065	.072	.071	.078	.080	.078	.071	.064
$t(4)$	.045	.032	.037	.043	.045	.047	.042	.040	.036
$t(5)$	.035	.045	.046	.044	.051	.062	.059	.052	.047

Table 8: Empirical power of the test for MAR(1,1) estimated by GCov (1000 replications) at 5% from samples of $T=400$. Each row corresponds to an error distribution: $t(3)$, $t(4)$, $t(5)$. Columns give the rejection frequency (Power) for different values of the noncausal autoregressive coefficient ψ (from 0.1 to 0.9). The statistic is evaluated at the observation corresponding to the 210th largest value among 400 simulated observations (52.5th percentile).

$\psi = 0.6$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.730	.775	.783	.795	.800	.790	.781	.764	.759
$t(4)$	.730	.785	.757	.740	.756	.767	.756	.738	.724
$t(5)$	.660	.695	.707	.715	.720	.713	.707	.690	.681
$\psi = 0.7$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.740	.785	.793	.797	.802	.802	.791	.769	.747
$t(4)$	.750	.750	.737	.738	.740	.763	.759	.744	.726
$t(5)$	.800	.775	.763	.765	.760	.762	.759	.746	.723
$\psi = 0.8$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.750	.795	.793	.782	.764	.763	.761	.738	.714
$t(4)$	.800	.765	.753	.752	.756	.747	.739	.729	.709
$t(5)$	.740	.775	.753	.755	.752	.735	.737	.731	.700
$\psi = 0.9$									
Distribution	ϕ								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$t(3)$	.760	.725	.717	.713	.706	.698	.687	.676	.642
$t(4)$	.690	.720	.683	.677	.666	.670	.650	.636	.611
$t(5)$	.720	.730	.703	.688	.680	.647	.641	.625	.596

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