A Stochastic Tree with Application to Bubble Modelling and Pricing

C. Gourieroux [(1)], and J. Jasiak [(2)]

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[1]University of Toronto, Toulouse School of Economics and CREST.
[2]York University, Canada.

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Abstract

We introduce a new stochastic tree representation of a stationary submartingale process for modelling and pricing speculative bubbles on commodity and cryptocurrency markets. The model is compared to other trees proposed in the literature on bubble modelling and stochastic volatility approximation. We show that the proposed model is an extension of the well-known Blanchard-Watson bubble. The model provides (quasi) closed-form pricing formulas for the derivatives and variance swaps, which are derived and illustrated.

Keywords : Stochastic Tree, Stationary Submartingale, Speculative Bubble, Stochastic Intensity, Derivative Pricing, Variance Swap.

1 Introduction

The price processes of many financial assets display spikes and local explosions that can be interpreted as speculative bubbles. Such features are displayed by the time series of market indexes [Bates (2008), Phillips, Wu, Yu (2015), Gourieroux, Zakoian (2017)], commodity prices [Phillips, Yu (2011)], and exchange rates of cryptocurrencies such as the bitcoin/dollar rate [Gourieroux, Hencic (2015)]. The bubble phenomena in price processes have been studied in the literature on Periodically Collapsing Bubble (PCB) [see e.g. Taylor, Peel (1998)] and in recent articles on asymmetric cycles [see e.g. Gourieroux, Jasiak, Monfort (2016)]. Among the martingale models of price processes, the family of stationary (sub)martingales is of special interest, as these can accommodate price dynamics with recurrent speculative bubbles. Moreover, positive submartingale processes appear in the recent literature on present value as the asset price components, which are added to the so-called fundamental or forward solution with a smooth pattern [see e.g. Gourieroux, Jasiak, Monfort (2016)].

This paper introduces a new family of stationary submartingale processes. The proposed approach is suitable for financial applications, as i) the model is semi-parametric, hence flexible; ii) the asset price dynamics is represented as a binomial tree with stochastic intensities on the branches, which makes it comparable with other trees that approximates stochastic volatility in the literature on derivative pricing, for example; iii) the model leads to (quasi) closed-form formulas for nonlinear predictions at any horizon, allowing for option pricing; iv) its continuous time counterpart is a jump process with a stochastic drift, stochastic intensity and stochastic jump size. In this respect, the model arises as an alternative to the standard stochastic volatility models, for which the (quasi) closed-form formulas of derivative prices have been derived for some specific affine models [3].

This paper is organized as follows. The dynamic model is introduced in Section 2, where we discuss the property of stationary submartingale and compare the model with the literature on stochastic trees. Section 3 provides the quasi-closed form formulas of the term structure of nonlinear predictions and presents the pricing of bubble's derivatives. Next, the continuous time analogue of this stochastic tree is provided. Section 4 concludes. Proofs are

[3] See Fouque et al. (2000) for the attempt to find analytical formulas for option prices under stochastic volatility.

gathered in Appendices.

2 The dynamic model

The model is a stochastic parameter autoregressive process with two states of persistence determined by an indicator variable Z_t. When $Z_t = 0$, there is no persistence, and, when $Z_t = 1$, the persistence appears and leads to a local explosion. Therefore the trajectory of this process displays bubbles, but no global trend.

2.1 Stationary solution and moving average representation

Let us consider a strictly stationary process satisfying the autoregressive equation :

$$Y_t = \frac{\eta}{1-a} + \frac{1}{ap_t}(Y_{t-1} - \frac{\eta}{1-a})Z_t + \frac{\eta}{a(1-p_t)}(1-Z_t), \tag{2.1}$$

where a, η are two scalar parameters $a \in (0,1), \eta > 0$, and (p_t), (Z_t) are two processes such that :

• (p_t) is a strictly stationary process with values in $(0,1)$,

• (Z_t) is a strictly stationary process such that the variables $Z_t's$ are independent conditional on (p_t), with $Z_t \sim \mathcal{B}(1, p_t)$.

It follows that the bivariate process (p_t, Z_t) is jointly stationary.

Among the solutions of recursive equation (2.1), we focus on process (Y_t), that is measurable with respect to the filtration I_t generated by $p_t, Z_t, p_{t-1}, Z_{t-1}, \ldots$.

To do that, we solve recursively equation (2.1) to write Y_t as a function of Y_{t-H} and processes (p_t) and (Z_t) between $t-H+1$ and t, where $H \geq 1$. We get :

$$\begin{aligned} Y_t &= \frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{H-1}\left[\frac{1}{a^h}\frac{1}{(1-p_{t-h})\Pi_{k=0}^{h-1}p_{t-k}}(1-Z_{t-h})\Pi_{k=0}^{h-1}Z_{t-k}\right] \\ &+ \frac{1}{a^H}\frac{1}{\Pi_{k=0}^{H-1}p_{t-k}}\Pi_{k=0}^{H-1}Z_{t-k}(Y_{t-H} - \frac{\eta}{1-a}). \end{aligned} \tag{2.2}$$

This process has an infinite moving average representation given below, and it is strongly stationary under the following assumption :

Assumption A.1 : At any date t, the sum $\sum_{k=0}^{H} \log p_{t-k}$ tends a.s to $-\infty$, when H tends to ∞.

When the state intensity process (p_t) is ergodic, $\frac{1}{H}\sum_{k=0}^{H} \log p_{t-k}$ tends a.s. to the expectation $E \log p_t$ computed under its stationary distribution. If p_t is not equal to 1 a.s., $\forall t = 1, \dots,$ this expectation is negative and possibly equal to $-\infty$; then Assumption A.1 is satisfied.

Proposition 1 : Under Assumption A.1, there exists a unique strictly stationary solution to recursive equation (2.1). This solution admits the following (nonlinear) one-sided infinite moving average [MA (∞)] representation :

$$Y_t = \frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{\infty}\left[\frac{1}{a^h}\frac{1}{(1-p_{t-h})\Pi_{k=0}^{h-1}p_{t-k}}(1-Z_{t-h})\Pi_{k=0}^{h-1}Z_{t-k}\right]. \qquad (2.3)$$

Proof : See Appendix 1.1

The moving average representation (2.3) implies the following restriction on the domain of the stationary distribution of (Y_t).

Corollary 1 : The stationary process (2.1) is such that $Y_t \geq \frac{\eta}{1-a}$.

Alternatively, the existence of a stationary solution of model (2.1) can be shown as follows. Model (2.1) can be rewritten as a linear affine autoregression with stochastic parameters :

$$Y_t - \frac{\eta}{1-a} = \xi_{1t}\left(Y_{t-1} - \frac{\eta}{1-a}\right) + \xi_{2t}, \qquad (2.4)$$

where $\xi_{1t} = \frac{1}{a}\frac{Z_t}{p_t}$ is the stochastic autoregressive parameter and the stochastic drift is $\xi_{2t} = \frac{\eta}{a}\frac{1-Z_t}{1-p_t}$.

In this stochastic linear autoregression, the coefficient process $\xi_t = (\xi_{1t}, \xi_{2t})'$ is strictly stationary. For a strictly stationary and ergodic (ξ_t), Brandt (1986) derived sufficient conditions for the existence and uniqueness of a stationary solution to dynamic equation (2.4). It is easy to check that the sufficient conditions given in Brandt (1986) are satisfied by model (2.1) as well.

2.2 A stationary submartingale

Process (Y_t) is an example of a stationary positive submartingale with respect to the filtration I_t, which includes $(\underline{Y_t}) = (Y_t, Y_{t-1}, \ldots)$. Process (Y_t) satisfies the submartingale condition given below :

Proposition 2 : The process (Y_t) is such that :

$$E(Y_t | I_{t-1}) = \frac{1}{a} Y_{t-1}.$$

Proof : See Appendix 1.2.

Remark 1 : Proposition 2 holds also when the information set is enlarged and includes not only $\underline{Y_t}$, but also the current and lagged values of process (p_t), that is when $I_t = \left(\underline{Y_t}, \underline{p_t}\right)$. Such an enlarged information set is used for pricing purpose when the investor is assumed to be more informed than the econometrician.

It follows that the submartingale process (Y_t) explodes in the conditional mean, when H tends to infinity :

$$\lim_{H \to \infty} E(Y_{t+H} | I_t) = \frac{1}{a^H} Y_t \to +\infty,$$

so that the process remains stationarity while displaying local short-lived explosions . Moreover Y_t is not integrable, i.e. its marginal mean does not exist [see Gourieroux, Jasiak, Monfort (2016)] :

Proposition 3 : The positive process (Y_t) is not integrable : $EY_t = +\infty$.

Proof : See Appendix 1.3.

The positive stationary submartingale can be interpreted as an extension of the bubble process given in Blanchard, Watson (1982). More specifically, the latter process is a special case of (Y_t) with constant intensity process $p_t = p$ and $\eta = 0$. In this special case Assumption A.1 is clearly satisfied:

$$\sum_{h=0}^{H} \log p_{t-h} = (H+1)\log p \text{ tends to } -\infty, \text{if } p < 1.$$

Remark 2 : Positive submartingales with $E(Y_t|\underline{Y_{t-1}}) = \frac{1}{a}Y_{t-1}$, for $0 < a < 1$ considered in the literature have either trajectories that tend to $+\infty$ when t tends to infinity, or to zero [see e.g. Kamihigashi (2011) for the discussion of the so-called explosive and implosive bubbles]. The process (Y_t) in (2.1) is neither explosive, nor implosive, and has no global trend. Its trajectories feature periodically collapsing bubbles (PCB) as shown in the simulations below. So far, only two PCB models were used in the economic literature on rational expectation [see e.g. Charemza, Deadman (1995), Taylor, Peel (1998), Psaradakis et al. (2001), Phillips, Wu, Yu (2011)]. The first PCB model is the bubble introduced by Blanchard, Watson (1982). The second PCB model has been introduced by Evans (1991), as an extension of the Blanchard, Watson model. As it cannot be interpreted as a tree, its application to derivative pricing is limited. In addition, its stationarity conditions are unknown and there exists no (quasi) closed-form formula of the term structure of nonlinear predictions for that model.

Remark 3 : The process (Y_t) is undefined in the special case $a = 1$ corresponding to a (stationary) martingale. It can be transformed into a martingale (Y_t^*) by considering $Y_t^* = a^t Y_t$, for $a < 1$. However martingale (Y_t^*) is not stationary and tends to zero when t tends to infinity.

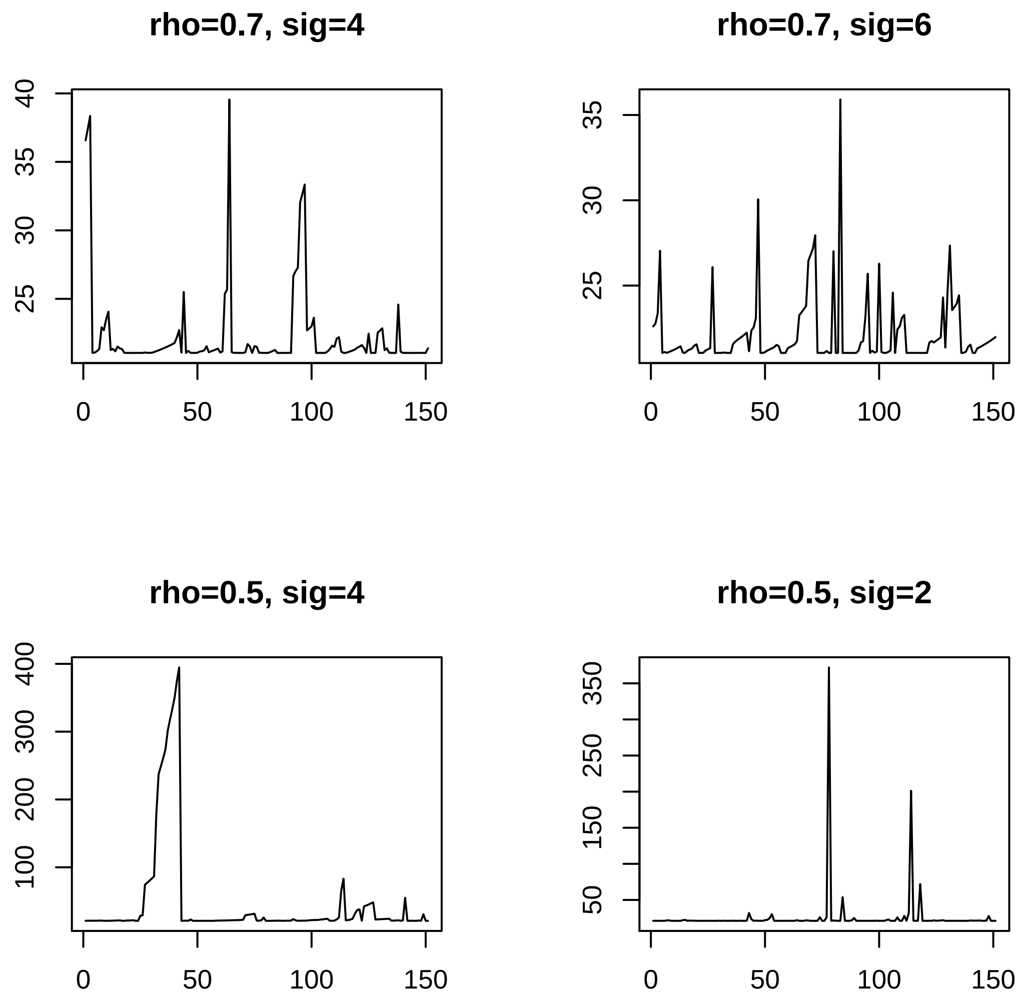
Let us now illustrate the patterns of trajectories of process (Y_t). In this illustration, the stochastic intensity is defined by $p_t = \Phi(X_t)$, where Φ is the cumulative distribution function of the standard normal and (X_t) is a latent stationary Gaussian autoregressive process such that :

$$X_t = \mu + \rho(X_{t-1} - \mu) + \sigma\sqrt{1-\rho^2}u_t,$$

where the errors $u_t's$ are i.i.d. standard normal, σ is positive and ρ strictly between -1 and 1. In particular, when $\mu = 0$ and $\sigma = 1$, the stationary distribution of X_t is standard normal and p_t is the theoretical Gaussian rank of X_t value. Let us choose an initial value $X_0 = \mu$, and set the parameters equal to : $\mu = -1, \eta = 1, a = 0.95$. The two remaining parameters are allowed to vary and take the following four sets of values : $\sigma = 4, \rho = 0.7$;

$\sigma = 6, \rho = 0.7, \sigma = 4, \rho = 0.5; \sigma = 2, \rho = 0.5$. By increasing the value of ρ, we increase the persistence[4] of p_t and create more extreme values of p_t in the sense that p_t is approaching 1. This in turn creates an explosive stochastic drift ξ_{2t}. Figure 1 shows the dynamics of (Y_t) with the above four sets of parameter values.

Figure 1 : Simulated Paths of Process Y



[4]This is an intensity clustering effect analogous to the volatility clustering.

The values of a and μ are used to generate high values of the stochastic autoregressive coefficient $\frac{1}{a}\frac{Z_t}{p_t}$ in the growth phase of a bubble. The ρ coefficient manages the persistence in the growth phase, whereas the σ parameter has a direct effect on the frequency of bubbles.

2.3 Comparison with the literature on stochastic trees.

The model (2.1) can be interpreted as a binomial tree in discrete time that has stochastic branches instead of deterministic branches as in the tree introduced by Cox, Ross, Rubinstein (1979). Let us describe the branching that starts at time t and stems from Y_{t-1}. The first branch, $Z_t = 1$, appears with (stochastic) intensity p_t so that the next value of the process on that branch is $Y_t = Y_{t-1}/(ap_t)$. On this branch Y_t follows an autoregression with an explosive(stochastic) autoregressive coefficient ξ_{1t}. The second branch : $Z_t = 0$, is generated with (stochastic) intensity $1 - p_t$ and the process becomes $Y_t = \frac{\eta}{1-a} + \frac{\eta}{a(1-p_t)}$, which is independent of Y_{t-1}. The standard interpretation of branches in terms of up/down movements as in the Cox, Ross, Rubinstein tree does not apply directly. The first branch is an up branch, since $Y_t = Y_{t-1}/(ap_t) > Y_{t-1}$ with a positive increment. The second branch creates either an up, or a down movement. More precisely, an up movement is observed if and only if $\frac{\eta}{a(1-p_t)} > Y_{t-1} - \frac{\eta}{1-a}$. This arises in two situations : i) when Y_{t-1} is close to $\eta/(1-a)$, which is a reflection of Y_t from the lower bound, and ii) for large Y_{t-1} and p_t close to 1, it accelerates the bubble growth. This dynamics can be related to the "volatility induced mean reversion" when very small (or very large) values of Y_t make the process bounce off the lower bound.

In the literature, stochastic trees provide tractable approximations of continuous time diffusion models by discretizing both time and space. Among the examples are the Cox, Ross, Rubinstein binomial tree that approximates the Black, Scholes diffusion [Cox, Ross, Rubinstein (1979)], the extension considered by Nelson, Ramaswamy (1990) to approximate a more general diffusion $dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t$, which consists in preliminary transforming the process into a diffusion process with constant volatility. The stochastic tree (2.1) is of a different type as it depends on two latent random processes (p_t) and (Z_t), respectively. It is comparable to trees that approximate continuous time stochastic volatility models [see e.g. Gruber, Schweizer (2006), Florescu, Viens (2005), (2008), Akyldirim, Dolinsky, Mete Soner (2014)]. The difference is in the stochastic volatility being replaced by the stochastic intensity. Note that stochastic tree (2.1) is based on two dependent state variables either (p_t, Z_t), or (ξ_{1t}, ξ_{2t}) [see e.g. Hilliard, Schwartz (1996) for such correlated state variables in the stochastic volatility framework].

To facilitate the comparison with the literature on trees with finite state space conditional on the past, we study a special case of (Y_t) that arises when the intensity process (p_t) in (2.1) is a Markov chain with two states $\bar{p}_0, \bar{p}_1$, say, and a transition matrix $\begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}$. Then, for lagged intensity $p_{t-1} = \bar{p}_0$, we get a tree with four branches (quadrinomial tree) as in the scheme below :

Scheme : A quadrinomial tree

branch	p_t	Z_t	probability of the branch	future value
$0,1$	$\bar{p}_0$	1	$\pi_{00}\bar{p}_0$	$\eta/(1-a) + (1/a\bar{p}_0)[Y_{t-1} - \eta/(1-a)]$
$0,0$	$\bar{p}_0$	0	$\pi_{00}(1-\bar{p}_0)$	$\eta/(1-a) + \eta/[a(1-\bar{p}_0)]$
$1,1$	$\bar{p}_1$	1	$\pi_{01}\bar{p}_1$	$\eta/(1-a) + (1/a\bar{p}_1)[Y_{t-1} - \eta/(1-a)]$
$1,0$	$\bar{p}_1$	0	$\pi_{01}(1-\bar{p}_1)$	$\eta/(1-a) + \eta/[a(1-\bar{p}_1)]$

For a different value of lagged intensity $p_{t-1} = \bar{p}_1$, the same future values can occur with different probabilities. The lower bound for the trajectories of (Y_t) is : $\eta/(1-a) + \min\{\eta/[a(1-\bar{p}_0)], \eta/[(1-\bar{p}_1)]\}$, which is strictly larger than $\eta/(1-a)$.

3 Nonlinear prediction

This section presents closed-form nonlinear prediction formulas for $Y_{t+H}, H \geq 1$. Next, these prediction formulas are used to obtain new (quasi) closed-form pricing formulas for derivatives written on the bubble component of Y_t when the dynamics is considered under a risk-neutral probability. Under a constant (continuously compounded) riskfree rate r, the absence of arbitrage opportunity implies that the price of the underlying asset Y_t satisfies the condition :

$$Y_{t-1} = \exp(-r) E_{t-1} Y_t, \tag{3.1}$$

under a risk-neutral probability. This condition is equivalent to the submartingale condition in Proposition 2 with $a = \exp(-r)$, and $r > 0$, as the discount factor.

The pricing formula (3.1) can be compared with the pricing formulas derived from the Hull, White stochastic volatility model [Hull, White (1987), Ball, Roma (1994)], for instance. In a discrete time model, the market is incomplete and the submartingale condition (3.1) provides no information on the historical dynamics of the price process. However in the limiting case when the risk-neutral and historical distributions coincide, the historical dynamics (2.1) features PCB and we price the effects of these PCBs on the derivative payoffs. More general results are obtained in the same semi-parametric family of dynamic models with different parameters [η, a, distribution of (p_t)] for the historical and risk-neutral distributions. Then, under the absence of arbitrage opportunity, these distributions have the same support in the two worlds (see Appendix 1.1 for the form of the support).

3.1 Term structure of nonlinear predictions

The autoregressive representation (2.1) allows us to derive nonlinear predictions of the process at any horizon $H \geq 1$. This is done by means of the conditional moment generating function of the positive process $Y_t - \dfrac{\eta}{1-a}$, or equivalently by means of the conditional real Laplace transform of the process $\log(Y_t - \dfrac{\eta}{1-a})$. All these computations are performed conditional on the enlarged information set $I_t = (\underline{Y_t}, \underline{p_t})$, which is the information set of the investor (see Remark 1).

Proposition 4 : The Laplace transform is :

$$
\begin{aligned}
& E_t\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^{\alpha}\right] \\
= & \sum_{h=0}^{H-1} \left\{ \frac{\eta^{\alpha}}{a^{\alpha(h+1)}} E_t\left(\left[(1-p_{t+H-h})\Pi_{k=0}^{h-1} p_{t+H-k}\right]^{1-\alpha}\right)\right. \\
+ & \frac{1}{a^{\alpha H}}\left(Y_t - \frac{\eta}{1-a}\right)^{\alpha} E_t\left[\left(\Pi_{k=0}^{H-1} p_{t+H-k}\right)^{1-\alpha}\right],
\end{aligned}
$$

where E_t denotes the expectation conditional on information $I_t = (\underline{Y_t}, \underline{p_t})$ and α is any nonnegative scalar such that the conditional expectations :

$$E_t[(1-p_{t+H-h})^{1-\alpha}(\Pi_{k=0}^{h-1} p_{t+H-k})^{1-\alpha}] \text{ exist for } h = 0, \ldots, H-1,$$

as well as :$E_t[(\Pi_{k=0}^{H-1} p_{t+H-k})^{1-\alpha}]$.

Proof : See Appendix 1.4.

Thus, the conditional prediction of $\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha$ is a linear affine function of $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$, with coefficients depending on the current and lagged values of the process (p_t) of stochastic intensities, since (Y_t) does not Granger cause (p_t). The prediction formulas are given below for various intensity processes (p_t).

a) i.i.d. stochastic intensities

The nonlinear prediction formula is greatly simplified, when the stochastic intensities p_t are independent and identically distributed (i.i.d.).

Corollary 2 : Let us assume that the stochastic intensities (p_t) are i.i.d.. Then,

$$\begin{aligned} & E_t\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha\right] \\ = \quad & c(\alpha)\frac{\eta^\alpha}{a^\alpha}\frac{1-\dfrac{[\psi(1-\alpha)]^H}{a^{\alpha H}}}{1-\dfrac{\psi(1-\alpha)}{a^\alpha}} + \left[\frac{\psi(1-\alpha)}{a^\alpha}\right]^H \left(Y_t - \frac{\eta}{1-a}\right)^\alpha, \end{aligned}$$

where :

$$c(\alpha) = E[(1-p_t)^{1-\alpha}], \quad \psi(\alpha) = E(p_t^\alpha),$$

and α is such that $c(\alpha)$ and $\psi(1-\alpha)$ exist. [5]

[5] The existence is insured for α between 0 and 1.

In the framework of i.i.d. stochastic intensities, we get a term structure of nonlinear predictions of the type :

$$E_t\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}\right]=d_0(H,\alpha)+d_1(H,\alpha)\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha},\forall H,$$

where coefficients d_0, d_1 are deterministic functions of term H. Thus, the long run behavior of the term structure of predictions depends on the choice of the common distribution of stochastic intensities (p_t) and on the power coefficient α.

Corollary 3 : Under the i.i.d. stochastic intensities(p_t), the long run prediction $\lim_{H\to\infty} E_t\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}\right]$ exists (and is independent of Y_t) if $\psi(1-\alpha)/a^{\alpha}<1$. Otherwise, that limit is infinite.

As mentioned earlier, if $\alpha=1$, then $\psi(1-\alpha)/a^{\alpha}=1/a>1$ and hence the expectation $\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha}$ is explosive. However, for α sufficiently small, we can expect the marginal expectation $\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha}$ to be finite.

Let us now illustrate these results by examples.

Example 1 : Constant intensities

If $p_t=p$ is constant, we get $c(\alpha)=(1-p)^{1-\alpha}$, and $\psi(1-\alpha)=p^{1-\alpha}$. The condition for the convergence of the series in Corollary 2 is : $p^{1-\alpha}/a^{\alpha}<1$, which is equivalent to $\alpha<\log p/\log(ap)$. Since $\log p/\log(ap)=\log p/(\log a+\log p)<1$, we see that Y_t is non integrable whereas the conditional expectation of $\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha}$ exists and is finite for any α sufficiently small.

Example 2 : Uniform stochastic intensities

When the stochastic intensities (p_t) follow a uniform distribution on (0,1), we get : $c(\alpha)=E[(1-p_t)^{1-\alpha}]=E(p_t^{1-\alpha})=\psi(1-\alpha)=\frac{1}{2-\alpha}$. Therefore the conditional expectation of $\left(Y_t-\frac{\eta}{1-\alpha}\right)^{\alpha}$ exists for $\alpha<2$. For $\alpha<2$, the condition for convergence of the long run prediction becomes $\psi(1-\alpha)/a^\alpha = \frac{1}{(2-\alpha)a^\alpha} < 1$.

Example 3 : Stochastic intensities with log-gamma distributions

Let us now assume that $-\log p_t$ follows a gamma distribution $\gamma(\nu)$ with degree of freedom ν. We get :

$$\begin{aligned}\psi(1-\alpha) &= E(p_t^{1-\alpha}) = E(\exp[(-\log p_t)(\alpha-1)]) \\ &= \int_0^\infty \exp[(\alpha-1)z]\frac{\exp(-z)z^{\nu-1}}{\Gamma(\nu)}dz \\ &= \int_0^\infty \exp[-(2-\alpha)z]\frac{z^{\nu-1}}{\Gamma(\nu)}dz = \frac{1}{(2-\alpha)^\nu},\end{aligned}$$

which exists for $\alpha < 2$.

The condition of convergence of long run predictions becomes $\frac{1}{(2-\alpha)^\nu a^\alpha} < 1$. As a limiting case, this example includes the uniform stochastic intensities of Example 2 for $\nu = 1$.

Example 4 : Beta stochastic intensities

When the intensities (p_t) follow a beta (β, γ) distribution with probability density function (p.d.f.) : $f(p) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}p^{\beta-1}(1-p)^{\gamma-1}$, with $\beta > 0, \gamma > 0$, we get :

$$\begin{aligned}\psi(1-\alpha) &= E(p^{1-\alpha}) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta+\gamma-\alpha+1)}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}, \\ \text{and} \quad c(\alpha) &= E[(1-p)^{1-\alpha}] = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta+\gamma-\alpha+1)}\frac{\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma)}.\end{aligned}$$

The conditional expectation of $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$ exists, if $\alpha < \min(\beta, \gamma)+1$.

b) Compound Autoregressive log-intensity process

Let us now assume that the (negative) log-intensity process ($\log p_t$) is a compound autoregressive (CaR) process[6] [Darolles, Gourieroux, Jasiak (2006)]. This process is a Markov process with a conditional Laplace transform which is an exponential affine function of the conditioning value :

$$E[\exp(u \log p_{t+1})|p_t] = \exp[a(u) \log p_t + b(u)], \text{ say,} \tag{3.2}$$

where the argument u is nonnegative. Then the affine property of the log-Laplace transform is satisfied at any horizon h and it also holds for the cumulated process, that is,

$$E[\exp u[\log p_{t+1} + \ldots + \log p_{t+h}]|p_t] \equiv \exp[A(h,u) \log p_t + B(h,u)], \tag{3.3}$$

where functions $A(h,u), B(h,u)$ are easily derived recursively. For such a process, the conditional expectation in the prediction formula of Proposition 4 can be simplified. For horizons $h < H$, we get :

$$\begin{aligned} & E_t[(1-p_{t+H-h})^{1-\alpha}(\Pi_{k=0}^{h-1} p_{t+H-k})^{1-\alpha})] \\ = \ & E_t\{(1-p_{t+H-h})^{1-\alpha} p_{t+H-h}^{A(h,1-\alpha)}\} \exp B(h, 1-\alpha), \end{aligned}$$

and for $h = H$, we have:

$$E_t[(\Pi_{k=0}^{H-1} p_{t+H-k})^{1-\alpha}] = p_t^{A(H,1-\alpha)} \exp B(H, 1-\alpha).$$

The prediction formulas for $h < H$ involve only one-dimensional integrals and are easy to compute, while the above prediction formula for $h = H$ has a closed form.

Example 5 : Autoregressive Gamma process

This process is the time discretized Cox, Ingersoll, Ross process [Cox, Ingersoll, Ross (1985)]. The short-term conditional Laplace transform is [Gourieroux, Jasiak (2006)] :

$$E_t[\exp(u \log p_{t+1})] = \exp\left[\frac{u}{1+u} \log p_t - \delta \log(1+u)\right],$$

[6]The CaR processes are the discrete time analogues of affine processes considered in continuous time [Duffie, Filipovic, Schachermayer (2005)].

where δ is a parameter, $a(u) = \frac{u}{1+u}$, and $b(u) = -\delta \log(1+u)$. This provides closed-form prediction formulas.

3.2 Derivative pricing

Let us now consider derivatives written on a bubble [7](asset)$Y_t, Y_t > 0$. Under the absence of arbitrage opportunity and the assumption of constant(continuously compounded) riskfree rate r, $r > 0$, there exists a risk-neutral distribution Q such that :

$$E_t^Q(Y_{t+1}) = \exp(r)Y_t. \tag{3.4}$$

This is a submartingale condition with $a = \exp(-r) < 1$. Then, we can consider the derivative pricing formula when the price process (Y_t) is stationary and has risk-neutral dynamics (2.1). The price at date t of a European derivative paying $\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha$ at date $t+H$ is equal to $\exp(-rH)E_t \left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha\right] = a^H E_t \left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha$, and this price follows directly from Proposition 4.

The risk-neutral dynamics of (Y_t) cannot be chosen arbitrarily, as the current price of the underlying asset needs to exist and be finite. For instance in the case of $\alpha = 1$, the price has to exist (see the remark after Corollary 3). When the conditional expectation does not exist, some derivative payoffs cannot be traded on the derivative market, because they are too expensive and/or too costly to hedge. Therefore they are non-insurable. This is especially important given that we are interested in the pricing/hedging of rare and extreme spikes associated with periodically collapsing speculative bubbles. As another example, let us consider variance swaps for $\alpha = 2$. If such variance swaps are traded with an underlying asset value featuring bubbles, the models in Examples 2-3 cannot be used for pricing, whereas models in Examples 1 and 4 can, with $\min(\beta, \gamma) > 1$ in the latter case.

Example 6 : Variance Swaps Let us consider i.i.d. stochastic probabilities with finite conditional expectation for $\alpha = 2$ (see e.g. Example 4).

[7]A bubble is an asset whose price can significantly differ from its fundamental value [see, Tirole (1985)].

From Corollary 2 and the submartingale condition, it follows that the price of a historical volatility[8] paying Y_{t+H}^2 at date $t+H$ is :

$$V(Y_t,H) \equiv 2\frac{\eta}{1-a}Y_t - \frac{\eta^2 a^H}{(1-a)^2} + c(2)\eta^2 a^{H-2}\frac{1-\frac{\psi(-1)^H}{a^{2H}}}{1-\frac{\psi(-1)}{a^2}} + \frac{\psi(-1)^H}{a^H}\left(Y_t - \frac{\eta}{1-a}\right)^2. \tag{3.5}$$

This formula can be compared with the price of historical variance derived from the Black-Scholes formula with volatility parameter σ and continuous compounding parameter $a = \exp(-r)$. We get :

$$V^{BS}(Y_t,H) = \frac{\exp(\sigma^2 H)}{a^H}Y_t^2. \tag{3.6}$$

The proposed formula (3.5) for $V(t,H)$ appears more complex and also more flexible than the Black-Scholes formula, as it involves a "scale" parameter η, and two stochastic intensity parameters: parameter $\psi(-1)$, which is larger or equal to 1, and parameter c(2), which is also larger or equal to 1. Moreover it depends on the past and current values through an affine combination of Y_t and Y_t^2 instead of Y_t^2 only as in $V^{BS}(Y_t,H)$.

For a large current value of Y_t, the two formulas are equivalent when $\psi(-1) = \exp(\sigma^2)$. Thus, the term structures of variance prices are equivalent at infrequent times when the bubble takes extreme values (see Figure 1). Otherwise, the formulas differ necessarily, as the Black-Scholes model does not account for the possibility of extreme bubble patterns.

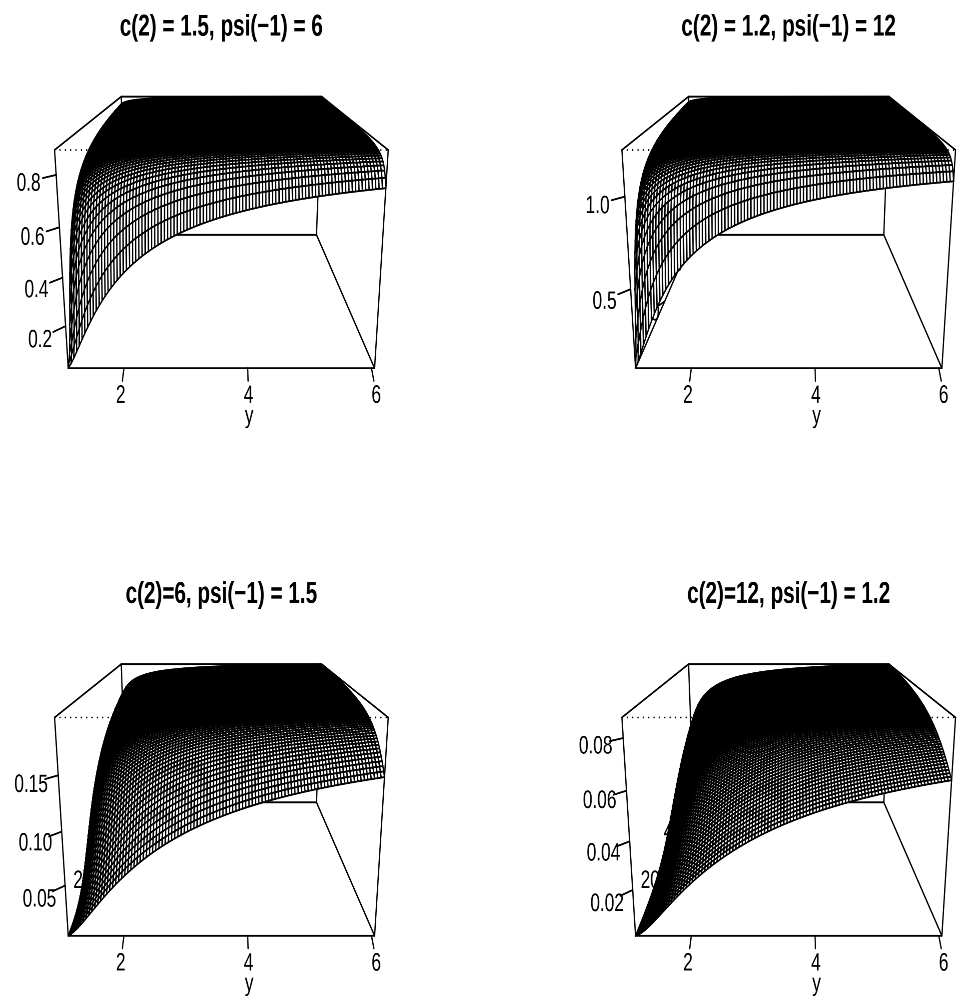
By analogy with the standard practice of computing Black-Scholes implied volatility, we can compute the quantity :

$$\sigma^{BS}(Y_t,H) = \sqrt{\frac{1}{H}\log\left[\frac{V(Y_t,H)a^H}{Y_t^2}\right]}, \tag{3.7}$$

obtained by inverting the Black-Scholes formula for historical variance (3.6) with the value of the bubble derivative substituted for $V^{BS}(Y_t,H)$. Figure 2 provides the Black-Scholes implied volatility surfaces for H and $Y_t = y$ varying, and the parameters $\eta, a, c(2), \psi(-1)$ held fixed at constant values.

[8]Such historical volatility is the key component of a variance swap.

Figure 2 : Implied BS Volatility Surfaces



The parameters $c(2) = E[1/(1-p)]$ and $\psi(-1) = E(1/p)$ are constrained by the inequality given below. For instance, since $\dfrac{1}{1-p} = 1 + \dfrac{1}{1/p - 1}$, it follows from the Jensen's inequality that :

$$c(2) = E[1/(1-p)] = 1 + E[1/(\frac{1}{p} - 1)] \geq 1 + \frac{1}{E(1/p) - 1} = \frac{\psi(-1)}{\psi(-1) - 1}.$$

Similarly, we could derive an upper bound for $c(2)$ for a given $\psi(-1)$. To satisfy such inequality constraints, we consider beta stochastic intensities (see Example 4). We get :

$$c(2) = (\beta + \gamma - 1)/(\gamma - 1), \psi(-1) = (\beta + \gamma - 1)/(\beta - 1),$$

where β and γ are larger than 1 to satisfy the condition of existence of the conditional expectation.

The values of the parameters take these constraints into account. The values of $\eta = 0.01$ and $a = 0.99$ are fixed that implies a lower bound equal to 1 for process (Y_t). The selected pairs for $[c(2), \psi(-1)]$ (or equivalently for (β, γ)) are :

$$\begin{array}{ll} \beta = 2,\ \gamma = 5 & \text{yielding } c(2) = 1.5, \psi(-1) = 6; \\ \beta = 2,\ \gamma = 11 & \text{yielding } c(2) = 1.2, \psi(-1) = 12; \\ \beta = 5,\ \gamma = 2 & \text{yielding } c(2) = 6, \psi(-1) = 1.5; \\ \beta = 11, \gamma = 5 & \text{yielding } c(2) = 12, \psi(-1) = 1.2. \end{array}$$

The two first (resp. second) scenarios correspond to infrequent bubbles with large rates of explosion (resp. frequent bubbles with small rates of explosion).

Figure 2 shows that for small and medium values y, the price of volatility swap is higher than the limiting Black-Scholes price. Thus, the model may adjust for miss-pricing observed on the option markets, in practice. As noted in Coval, Shumway (2001), Bakshi, Kapadia (2003), the current market is "generating surprising large returns from selling crash insurance via out-of-the money put options".

3.3 Continuous time analogue

The literature on derivative pricing examines the relationship between the discrete and continuous time models to find out if a sequence of discrete time models at increasing frequencies tends to a well-defined continuous time model [see e.g. Stroock, Varadhan (1979)]. For a Markov process (Y_t), this can be done by considering the infinitesimal generator : $Gf(y) = \lim_{dt \to 0} \frac{1}{dt} E[f(Y_{t+dt}) - f(Y_t) | Y_t = y]$. Since the intensity process has exogenous dynamics, the infinitesimal generator is derived conditional on the intensity process (p_t).

In Appendix 2 we find that the limiting continuous time dynamics of (Y_t) can be written as :

$$
\begin{aligned}
dY_t &= (\alpha+\pi_t)(Y_t-\frac{\eta}{\alpha})dt+[\eta(\frac{1}{\alpha}+\frac{1}{\pi_t})-Y_t]dN_t \qquad (3.8)\\
&= \alpha Y_t dt+[\eta(\frac{1}{\alpha}+\frac{1}{\pi_t})-Y_t](dN_t-\pi_t dt), \qquad (3.9)
\end{aligned}
$$

where (N_t) is a counting process with stochastic intensity $\pi_t, \pi_t > 0$. Parameters α, η and process (π_t) are the infinitesimal analogues of parameters a, η and of the process of probabilities (p_t). Thus, conditional on the exogenous process of stochastic intensities, (Y_t) is a jump process with a predetermined drift and jump magnitude, both functions of y_t and π_t. Model (3.8)-(3.9) needs to be completed by a specification for the exogenous dynamics of stochastic intensity (π_t). For example, when (π_t) follows a Cox, Ingersoll, Ross diffusion a continuous time analogue of Example 5 is obtained :

$$d\pi_t = (b-c\pi_t)dt+\sigma dW_t, \qquad (3.10)$$

where (W_t) is a Brownian motion, and parameters are such that $b>0, c>0, \sigma>0$, and $2b/\sigma^2<1$ to ensure the existence and stationarity of process (π_t).

The limiting jump dynamics of process (Y_t) was easy to anticipate. Note, however that model (3.8)-(3.9) differs from the model introduced in Bates (2008) for crash risk. More specifically, the model of Bates(2008) is :

$$dY_t \equiv \mu dt+\sigma dW_t+\gamma dN_t, \qquad (3.11)$$

where (W_t) is a Brownian motion, while (N_t) is a Poisson process with constant intensity λ, and μ, σ, γ are constant parameters. Like the Bates' model, model (3.8)-(3.9) is a two-factor model. It depends on more parameters, however. More specifically, the Bates' model has 3 parameters, while model (3.8)-(3.9) has 5 parameters. As mentioned earlier in the text, these additional parameters determine the frequency of the bubbles and the persistence of bubble during its growth.

The continuous time model (3.8)-(3.9) inherits some properties of the stochastic tree (2.1), such as the local drift, which is linear in Y_t and is a property of a positive submartingale.

4 Concluding Remarks

This paper introduces a new model for the dynamics of price processes with bubbles and a new approach for pricing of derivatives written on assets with bubble price processes. The new model has a tree representation, which makes it comparable to other trees for derivative pricing that exist in the literature. The model provides (quasi) closed-form pricing formulas that arise as alternatives to the Black-Scholes forlulas. Therefore, the proposed specification is a relevant addition to bubble models that include noncausal stationary processes [see, e.g. Gourieroux, Lu (2019) for bubble pricing based on noncausal processes].

For further research, there remains a question, if in practice, one can disentangle the local trends due to stationary bubbles, from the global trends due to nonstationary processes with random walk components? The answer to that question would clarify the need for pricing formulas that accommodate each of these patterns,as the prediction and pricing formulas need to be adjusted to the asset price dynamics. That also concerns the validity of test procedures for unit roots and martingales [see e.g. Phillips, Wu, Yu (2011), Phillips, Yu (2011), Gourieroux, Jasiak (2019) for discussion].

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Appendix 1

1.1 Proof of Proposition 1

i) Under Assumption A.1, we have :

$P[\Pi_{k=0}^{H-1} Z_{t-k} = 1 | (p_t)] = \Pi_{k=0}^{H-1} p_{t-k}$, which tends a.s to zero.

Since $\Pi_{k=0}^{H-1} Z_{t-k}$ takes only values 0 and 1, it follows that $\Pi_{k=0}^{H-1} Z_{t-k}$ is decreasing and tends a.s. to zero. Moreover, as (Y_t) is stationary, the last term in (2.2) tends a.s. to zero as well, when H tends to infinity. Therefore, a stationary solution of (2.1), if it exists, has the nonlinear moving average (MA) representation (2.3). In particular, it is unique.

ii) The MA (∞) representation (2.3) implies that, conditional on any given path of $(p_t), Y_t$ follows a discrete distribution and the values of Y_t :

$$\frac{\eta}{1-a} + \frac{\eta}{a} \frac{1}{a^h} \frac{1}{(1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k}},$$

have probabilities $(1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k}, h = 0, \ldots, \infty$. Indeed, under Assumption A.1, we can define the sum of these probabilities :
$\sum_{h=0}^{\infty} ((1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k})$, which is equal to 1. Therefore, the conditional distribution of Y_t given I_{t-1} exists and is independent of time t, which implies a homogeneous transition. Since the process (p_t) is strongly stationary and this transition is homogeneous, we deduce that the process (Y_t) itself is strictly stationary.

1.2 Proof of Proposition 2

From (2.1) we get :

$$
\begin{aligned}
E(Y_t|\underline{Y_{t-1}},\underline{p_t}) &= \frac{\eta}{1-a} + \frac{1}{ap_t}(Y_{t-1} - \frac{\eta}{1-a})E(Z_t|\underline{Y_{t-1}},\underline{p_t}) + \frac{\eta}{a(1-p_t)}[1 - E(Z_t|\underline{Y_{t-1}},\underline{p_t})] \\
&= \frac{\eta}{1-a} + \frac{1}{a}(Y_{t-1} - \frac{\eta}{1-a}) + \frac{\eta}{a} \\
&= \frac{1}{a}Y_{t-1} + \frac{\eta}{1-a} - \frac{1}{a}\frac{\eta}{1-a} + \frac{\eta}{a} \\
&= \frac{1}{a}Y_{t-1}.
\end{aligned}
$$

1.3 Proof of Proposition 3

We have :

$$
\begin{aligned}
EY_t &= EE[Y_t|p_t, p_{t-1}, \ldots] = E\left(\frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{\infty}\frac{1}{a^h}\right) \text{ (by (2.3))} \\
&= +\infty,
\end{aligned}
$$

since the geometric series $\sum_{k=0}^{\infty}\frac{1}{a^h}$ diverges for $a \in (0,1)$.

1.4 Proof of Proposition 4

It follows from (2.2) that :

$$
\begin{aligned}
(Y_{t+H} - \frac{\eta}{1-a})^{\alpha} &= \frac{\eta^{\alpha}}{a^{\alpha}}\sum_{h=0}^{H-1}\left\{\frac{1}{a^{\alpha h}}\frac{1}{[(1-p_{t+H-h})\Pi_{k=0}^{h-1}p_{t+H-k}]^{\alpha}}(1 - Z_{t+H-h})\right. \\
&\left.\Pi_{k=0}^{h-1}(1 - Z_{t+H-k})\right\} \\
&+ \frac{1}{a^{\alpha H}}\frac{1}{(\Pi_{k=0}^{H-1}p_{t+H-k})^{\alpha}}(\Pi_{k=0}^{H-1}Z_{t+H-k})(Y_t - \frac{\eta}{1-a})^{\alpha},
\end{aligned}
$$

by using the interpretation of the $Z_t's$ as indicator variables that implies $Z_t^{\alpha} = Z_t, (1-Z_t)^{\alpha} = 1 - Z_t$.

The result follows by noting that :

$$
\begin{aligned}
& E_t\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}\right] \\
= \;& E\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}|I_t\right] \\
= \;& E\left\{E\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}|I_t,\underline{p_{t+H}}\right]|I_t\right\}, \text{ by iterated expectations.}
\end{aligned}
$$

Appendix 2

Infinitesimal Generator

Let us consider the behavior of process Y at a small time step δ. The recursive equation (2.1), indexed by δ, becomes :

$$Y_{t+\delta}=\frac{\eta_\delta}{1-a_\delta}+\frac{1}{a_\delta p_{\delta t}}(Y_t-\frac{\eta_\delta}{1-a_\delta})Z_{\delta t}+\frac{\eta_\delta}{a_\delta(1-p_{\delta t})}(1-Z_{\delta t}), \qquad \text{(a.1)}$$

where $Z_{\delta t}\sim\mathcal{B}(1,p_{\delta t})$ and t is a multiple of δ. We assume that parameters and intensity process depend on δ as follows :

$$\eta_\delta=\eta\delta, a_\delta=\exp(-\alpha\delta), p_{\delta t}=\exp(-\pi_t\delta), \qquad \text{(a.2)}$$

where $\eta>0,\alpha>0$, and (π_t) is a positive process. π_t can be interpreted as the infinitesimal intensity on the branch without an autoregressive effect of Y_{t-1}. Then we have :

$$
\begin{aligned}
\frac{\eta_\delta}{1-a_\delta} &\sim \frac{\eta\delta}{\alpha\delta-\alpha^2\frac{\delta^2}{2}}=\frac{\eta}{\alpha(1-\frac{\alpha}{2}\delta)}=\frac{\eta}{\alpha}(1+\frac{\alpha}{2}\delta)=\frac{\eta}{\alpha}+\frac{\eta}{2}\delta, \\
\frac{1}{a_\delta} &\sim 1+\alpha\delta, \frac{\eta_\delta}{a_\delta}\sim\eta\delta.
\end{aligned}
$$

Therefore equation (a.1) becomes :

$$Y_{t+\delta} \sim \frac{\eta}{\alpha} + \frac{\eta\delta}{2} + (1+\alpha\delta)[Y_t - (\frac{\eta}{\alpha} + \frac{\eta\delta}{2})]\frac{Z_{\delta t}}{p_{\delta t}} + \eta\delta\frac{1-Z_{\delta t}}{1-p_{\delta t}}, \qquad \text{(a.3)}$$

where $Z_{\delta t}$ is Bernoulli distributed $B(1, p_{\delta t})$.

Since the process is nonnegative, its transition conditional on π_t is characterized by its conditional Laplace transform given below and written for a nonnegative argument u :

$$\begin{aligned}
\psi_{t,\delta}(u) &= E[\exp(-uY_{t+\delta})|Y_t, \pi_t] \\
&\sim \exp[-u(\frac{\eta}{\alpha} + \frac{\eta}{2}\delta)]\{p_{\delta t}\exp\{-u\frac{[1+\alpha\delta]}{p_{\delta t}}[Y_t - (\frac{\eta}{\alpha} + \frac{\eta}{2}\delta)]\} \\
&+ (1-p_{\delta t})\exp(-\frac{u\eta\delta}{1-p_{\delta t}})\} \\
&\sim \exp(-u\eta/\alpha)\exp(-\frac{u\eta\delta}{2})[(1-\pi_t\delta)\exp\{-u[1+(\alpha+\pi_t)\delta](Y_t - \frac{\eta}{\alpha} - \frac{\eta}{2}\delta)] \\
&+ \pi_t\delta\exp(\frac{-u\eta}{\pi_t})\} \\
&\sim \exp(-uY_t)(1-\frac{u\eta\delta}{2})(1-\pi_t\delta)[1-u\delta(\alpha+\pi_t)(Y_t - \frac{\eta}{\alpha}) + \frac{u\eta\delta}{2}] \\
&+ \pi_t\delta\exp[-u\eta(\frac{1}{\alpha} + \frac{1}{\pi_t})] \\
&\sim \exp(-uY_t)[1+\delta(-\pi_t - u(\alpha+\pi_t)(Y_t - \frac{\eta}{\alpha})] + \pi_t\delta\exp[-u\eta(\frac{1}{\alpha} + \frac{1}{\pi_t})].
\end{aligned}$$

It follows that :

$$\begin{aligned}
&\frac{1}{\delta}[\psi_{t,\delta}(u) - \psi_{t,0}(u)] \\
= \ &\frac{1}{\delta}E[\exp(-uY_{t+\delta}) - \exp(-uY_t)|Y_t, \pi_t] \\
\sim \ &-\exp(-uY_t)[\pi_t + u[(\alpha+\pi_t)(Y_t - \frac{\eta}{\alpha})]] + \exp[-u\eta(\frac{1}{\alpha} + \frac{1}{\pi_t})]\pi_t.
\end{aligned}$$

(a.4)

This formula provides the differential form of the infinitesimal generator for exponential transformations $f(y) = \exp(-uy), \forall u$. It also holds for any differentiable function f derived from these exponential transformations by linear combination and closure.

Thus, the necessary form of the infinitesimal generator of the Markov process (Y_t) conditional on process (π_t) is :

$$Gf(y) = \pi[-f(y) + f(\eta(\frac{1}{\alpha} + \frac{1}{\pi}))] + \frac{df(y)}{dy}[(\alpha + \pi)(y - \frac{\eta}{\alpha})].$$

This form is interpreted in terms of jump processes in Section 3.3.