A Stochastic Tree for Bubble Asset Modelling and Pricing

Christian Gourieroux*, and Joann Jasiak†

September 17, 2024

Abstract

We introduce a new stochastic tree representation of a strictly stationary submartingale process for modelling, forecasting and pricing speculative bubbles on commodity and cryptocurrency markets. The model is compared to other trees proposed in the literature on bubble asset modelling and stochastic volatility approximation. We show that the proposed model is an extension of the well-known Blanchard-Watson bubble. The model provides (quasi) closed-form pricing formulas for European options, which are derived and illustrated.

Keywords: Stochastic Tree, Stationary Submartingale, Bubble Pattern, Bubble Asset, Local Trends, Stochastic Intensity, European Option.

JEL Codes: C22, C53, C58

*University of Toronto, Toulouse School of Economics and CREST, e-mail: *Christian.Gourieroux @ensae.fr*

†York University, e-mail: *jasiakj@yorku.ca*

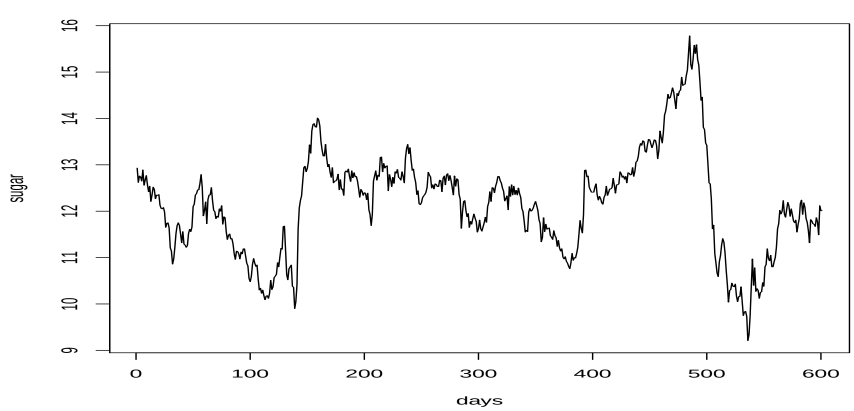
The authors gratefully acknowledge financial support of the chair ACPR : "Regulation and Systemic Risk", ERC DYSMOIA and Natural Sciences and Engineering Council of Canada (NSERC). We thank M. Schweizer and Y. Lu for helpful comments.

1 Introduction

The price processes of many financial assets display spikes and local explosions that can be interpreted as speculative bubbles manifesting themselves as price increases followed by sudden falls. Such bubble patterns are displayed by the time series of German hyperinflation over the period 1918-1924 [Psaradakis et al. (2001)], market indexes [Leybourne et al. (1996), Ofek and Richardson (2003), Pastor and Veronesi (2006), Bates (2008), Phillips, Wu, and Yu (2011), Phillips, Shi, and Yu (2015), Gourieroux and Zakoian (2017), Banerjee et al. (2020)], commodity prices [Phillips and Yu (2011), Gourieroux and Jasiak (2017), Cubadda, Hecq and Telg (2019), Hecq and Voisin (2021), (2023), Cubadda, Hecq and Voisin (2022), Gourieroux, Hencic and Jasiak (2020)], and exchange rates of cryptocurrencies such as the bitcoin/dollar rate [Gourieroux and Hencic (2015), Cavaliere, Nielsen and Rahbek (2020), Hou et al. (2020), Gourieroux, Jasiak and Tong (2021), Hall and Jasiak (2024), Gourieroux and Jasiak (2024)]. The bubble phenomena in price processes have been studied in the literature on Periodically Collapsing Bubble (PCB) [see e.g. Taylor, Peel (1998)] and in recent articles on asymmetric cycles [see e.g. Gourieroux, Jasiak, and Monfort (2020)]. Among the martingale models of price processes, the family of stationary (sub)martingales is of special interest, as these can accommodate both historical price dynamics with recurrent speculative bubbles and the associated risk-neutral dynamics.

A path of a stationary price process is displayed below for illustration. Figure 1 shows daily sugar futures price in US Dollars recorded between May 03, 2018 and July 22, 2020, with the settlement date of July 22, 2020. The data on Sugar 11 ICE Futures (SB=F) are provided on-line by Yahoo Finance Canada. The code SB=F indicates the raw sugar number 11 future contract with physical delivery in the country of origin.

Figure 1 : Sugar Futures Price



We observe bubble and spike patterns in the absence of global explosive trends in futures price processes of many other commodities. From the theoretical point of view, this dynamic can be explained because positive stationary (sub)martingale processes appear in dynamic equilibria of rational expectation models as price components added to the so-called fundamental or forward solution [Gourieroux, Jasiak, and Monfort (2020)]. This leads to the economic definition of a bubble asset: A bubble asset is an asset whose price can significantly differ from its fundamental value [Tirole (1985)]. In general, the path of a bubble asset includes a bubble pattern.

We focus our attention on the time series in discrete time, since "the majority of the empirical studies are based on models in discrete time with infinite horizons and the traded (bubble) assets have no terminal payoff at $t = \infty$" [see Jarrow, Potter and Shimbo (2006), (2010) for bubble definition in continuous time and Section 3.3].

A stationary martingale model is introduced in Gourieroux and Zakoian (2017), who study noncausal autoregressive processes with local explosive patterns[3]. More examples of stationary martingale models are given in Gourieroux, Jasiak (2019) who propose an extension of the unit root test to a test of the general martingale hypothesis.

This paper introduces a new family of stationary submartingale processes. The proposed approach is suitable for the analysis of bubble asset prices as i) the model is semi-parametric, hence flexible; ii) the asset price dynamic is represented by a binomial tree with stochastic intensities on the branches, which makes it comparable with other trees for stochastic volatility approximation in the literature on derivative pricing; iii) the model leads to (quasi) closed-form formulas of nonlinear predictions at any horizon, allowing for option pricing; iv) its continuous time counterpart is a jump process with a stochastic drift, stochastic intensity and stochastic jump size. In this respect, the model arises as an alternative to standard stochastic volatility models, among which only some specific affine models have (quasi) closed-form formulas of derivative prices[4].

This paper is organized as follows. The dynamic model is introduced in Section 2, where we discuss the stationary submartingale property and compare the model with the literature on stochastic trees. Section 3 provides the quasi-closed form formulas of the term structure of nonlinear predictions and presents the pricing of bubble's derivatives. Next, the continuous time analogue of this stochastic tree is introduced. Section 4 concludes. Proofs are gathered in Appendices 1 to 3.

---

[3]Other causal-noncausal models have been used for bubble modelling in the recent literature.

[4]See Fouque et al. (2000) for the attempt to find analytical formulas for option prices under stochastic volatility.

2 The dynamic model

We consider a strictly stationary process, which satisfies a stochastic autoregression with two states of serial persistence determined by a latent indicator variable Z_t. When $Z_t = 0$, there is no persistence, and, when $Z_t = 1$, the persistence appears and leads to a local explosion. Therefore the trajectory of this process displays a sequence of temporary bubbles (i.e. local trends), but no global trend.

2.1 Stationary solution and moving average representation

As mentioned earlier, the model represents specific price processes (commodity prices, for example) that are strictly stationary (sub)martingales. Therefore, they can feature local explosions (bubbles and crashes), but no global trends. This model can also be used to represent a bubble component of a price process $X_t = X_{f,t} + Y_t$, where the fundamental component $X_{f,t}$ and the bubble component Y_t are assumed independent. More generally, in a multivariate model $X_t = X_{f,t} + \beta Y_t$, the univariate component Y_t can be introduced to represent a systemic bubble factor. Then, vector β contains the sensitivities of the components of X_t to the systemic bubble factor.

Let us consider a strictly stationary process satisfying the autoregressive equation :

$$Y_t = \frac{\eta}{1-a} + \frac{1}{ap_t}(Y_{t-1} - \frac{\eta}{1-a})Z_t + \frac{\eta}{a(1-p_t)}(1-Z_t), \tag{2.1}$$

where a, η are two scalar parameters $a \in (0,1), \eta > 0$, and (p_t), (Z_t) are two latent processes such that :

- (p_t) is a strictly stationary process with values in $(0,1)$,
- (Z_t) is a strictly stationary process such that the variables $Z_t's$ are independent conditional on (p_t), with Bernoulli distribution $Z_t \sim \mathcal{B}(1, p_t)$.

It follows that the bivariate process (p_t, Z_t) is jointly stationary (but not necessarily i.i.d.).

Among the solutions of recursive equation (2.1), we focus on process (Y_t) that is measurable with respect to the filtration I_t generated by $p_t, Z_t, p_{t-1}, Z_{t-1}, \ldots$. To do that, we solve recursively equation (2.1), and write Y_t as a function of Y_{t-H} and processes (p_t) and (Z_t) between $t-H+1$ and t, where $H \geq 1$. We get :

$$\begin{aligned} Y_t &= \frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{H-1}\left[\frac{1}{a^h}\frac{1}{(1-p_{t-h})\Pi_{k=0}^{h-1}p_{t-k}}(1-Z_{t-h})\Pi_{k=0}^{h-1}Z_{t-k}\right] \\ &+ \frac{1}{a^H}\frac{1}{\Pi_{k=0}^{H-1}p_{t-k}}\Pi_{k=0}^{H-1}Z_{t-k}(Y_{t-H} - \frac{\eta}{1-a}). \end{aligned} \tag{2.2}$$

This process has an infinite nonlinear moving average representation given below and is strongly stationary under the following assumption :

Assumption A.1 : At any date t, the sum $\sum_{k=0}^{H} \log p_{t-k}$ tends a.s. to $-\infty$, when H tends to ∞.

When the state intensity process (p_t) is ergodic, $\frac{1}{H}\sum_{k=0}^{H} \log p_{t-k}$ tends a.s. to the expectation $E \log p_t$ computed with respect to its stationary distribution. If p_t is not equal to 1 a.s., $\forall t$, this expectation is negative and possibly equal to $-\infty$; then Assumption A.1 is satisfied.

Proposition 1 : Under Assumption A.1, there exists a unique strictly stationary solution to the recursive equation (2.1). This solution admits the following (nonlinear) one-sided infinite moving average [MA (∞)] representation in (p_t, Z_t):

$$Y_t = \frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{\infty}\left[\frac{1}{a^h}\frac{1}{(1-p_{t-h})\Pi_{k=0}^{h-1}p_{t-k}}(1-Z_{t-h})\Pi_{k=0}^{h-1}Z_{t-k}\right]. \tag{2.3}$$

Proof : See Appendix 1.1

The moving average representation (2.3) implies the following restriction on the domain of the stationary distribution of (Y_t).

Corollary 1 : The stationary process (2.1) is such that $Y_t \geq \frac{\eta}{1-a}$.

The above inequality shows that the price process (Y_t) cannot take negative values. Model (2.1) can be rewritten as a linear affine autoregression with stochastic parameters :

$$Y_t - \frac{\eta}{1-a} = \xi_{1t}\left(Y_{t-1} - \frac{\eta}{1-a}\right) + \xi_{2t}, \tag{2.4}$$

where $\xi_{1t} = \frac{1}{a}\frac{Z_t}{p_t}$ is the stochastic autoregressive parameter and $\xi_{2t} = \frac{\eta}{a}\frac{1-Z_t}{1-p_t}$ is the stochastic drift.

2.2 Comparison with Random Coefficient Autoregressive Model

It is interesting to compare our model with the literature on Random Coefficient Autoregressive (RCA) models [Nicholls and Quinn (1981 a,b),(1983), Hwang and Basawa (1998)], including some stochastic unit root models [see e.g. Leybourne et al. (1996)]. Indeed, model (2.1) can be rewritten under an RCA specification as:

$$Y_t = \xi_{1t} Y_{t-1} + \epsilon_t, \tag{2.5}$$

with

$$\varepsilon_t = \frac{\eta}{1-a}[1 - \frac{1}{a}\frac{Z_t}{p_t}] + \frac{\eta}{a}\frac{1-Z_t}{1-p_t}. \tag{2.6}$$

It is easy to see that $E\epsilon_t = 0$. Let us now compute the covariance between the two random coefficients. We get:

$$Cov(\xi_{1t}, \epsilon_t) = E(\xi_{1t}\epsilon_t) = \frac{\eta}{a(1-a)} - \frac{\eta}{a^2(1-a)} E(\frac{1}{p_t}) \neq 0,$$

since $Z_t^2 = Z_t$. Thus, the two stochastic coefficients are correlated. This contradicts the independence assumption introduced in the definition of a CAR process [see e.g. Quinn and Nicholls (1981), Assumption iii), p.186, Hwang and Basawa (1998), Section 2.1.1, Akharif and Hallin (2003), p. 676, Regis et al. (2022), section 4.1.1].

An extension of RCA to correlated stochastic coefficients, known as the General Random Coefficients Autoregressive (GRCA) model has also been considered in the literature [Conlisk (1974), (1976), Regis et al. (2022), Section 4.1]. This literature focuses on the second-order properties of a second-order stationary solution Y_t, assuming that $E(Y_t^2)$ is finite[5]. In our framework introduced to represent local explosions, the strictly stationary solution Y_t has no finite second-order moment (see, Proposition 3).

Let us now discuss Assumption A.1 in the context of the RCA and GRCA literatures. First, note that the stability conditions of the GRCA literature assume the existence of second-order moments, which is not required in our model.

In the stochastic linear autoregression (2.4), the coefficient process $\xi_t = (\xi_{1t}, \xi_{2t})'$ is strictly stationary. For a strictly stationary and ergodic (ξ_t), a sufficient condition for the existence and uniqueness of a stationary solution to dynamic equation (2.4) is derived in Brandt (1986), Theorem 1. It is easy to check that the sufficient condition given in Brandt (1986) is satisfied by model (2.1) as well. Moreover, this condition is necessary. This is due to the special form of a tree with the stochastic autoregressive coefficient that takes the value zero with probability $1 - p_t$. This implies that the coefficient of Y_{t-H} in equation (2.2) can take the value zero and then stay equal to zero with probability $1 - \prod_{k=0}^{H-1} p_{t-k}$. It not

[5]See also the second-order moment condition in the RCA model in Nicholls and Quinn (1981a), p.186, Assumption v).

only converges to zero at a fast, exponential rate, but it also reaches zero and stays at zero. More precisely, this coefficient takes the value $1/\prod_{k=0}^{H-1} p_{t-k}$ with probability $\prod_{k=1}^{H-1} p_{t-k}$, and zero, otherwise. Thus, the condition of Assumption A.1 is necessary and sufficient for the a.s. convergence of this coefficient to zero.

Let us now compare this condition with the standard Lyapunov condition [see, e.g. Berkes et al. (2009), Horvath abd Trapani (2019), p. 340]. More specifically, the standard Lyapunov exponent condition for our process:

$$E \log |\xi_{1t}| = E[log \frac{Z_t}{a\, p_t}] < 0,$$

is always satisfied when Z_t takes the value zero with a strictly positive probability, since $E[\log \frac{Z_t}{a\, p_t}] = -\infty < 0$ (see the discussion of Assumption A.1)[6]. If $p_t = 1$ a.s., the stochastic autoregressive coefficient is deterministic and equal to $1/a$. Then, the Lyapunov exponent condition becomes $\log(1/a) < 0$, that is $a > 1$. Since by the definition of the process $a \in (0, 1)$, this case is excluded.

2.3 A stationary submartingale

Process (Y_t) is an example of a stationary positive submartingale with respect to the filtration I_t, which includes $(\underline{Y_t}) = (Y_t, Y_{t-1}, \ldots)$. Process (Y_t) satisfies the submartingale condition given below :

Proposition 2 : The process (Y_t) is such that :

$$E(Y_{t+1}|I_t) = E(Y_{t+1}|\underline{Y}_t) = \frac{1}{a} Y_t.$$

Proof : See Appendix 1.2.

Proposition 2 holds when the information set is enlarged and includes not only $\underline{Y_t}$, but also the current and lagged values of process (p_t), i.e. when $I_t = \left(\underline{Y_t}, \underline{p_t}\right)$. Such an enlarged information set is used for deriving pricing formulas when the investor is assumed to be more informed than the econometrician (see Section 3.2).

It follows that the submartingale process (Y_t) explodes in the conditional mean, when H tends to infinity :

[6] In particular, the model contradicts the common idea that a bubble model has a stochastic autoregressive coefficient that "remains very concentrated and close to unity" [Banerjee et al. (2020), p.301].

$$\lim_{H\to\infty} E(Y_{t+H}|I_t) = \frac{1}{a^H} Y_t \to +\infty,$$

without "permanent" explosions" in trajectories, so that the process remains stationary while displaying local short-lived explosions . Moreover Y_t is not integrable, i.e. its marginal mean does not exist, and hence its second-order moment does not exist either [see Gourieroux, Jasiak and Monfort (2020)] :

Proposition 3 : The positive process (Y_t) is not integrable : $EY_t = +\infty$.

Proof : See Appendix 1.3.

The positive stationary submartingale can be interpreted as an extension of the bubble process given in Blanchard and Watson (1982). More specifically, the latter process is a limiting case of (Y_t) with constant intensity process $p_t = p$ and $\eta = 0$ [7]. In this limiting case Assumption A.1 is clearly satisfied since:

$$\sum_{h=0}^{H} \log p_{t-h} = (H+1)\log p \text{ tends to } -\infty, \text{if } p < 1.$$

In general, the positive submartingales with $E(Y_{t+1}|\underline{Y_t}) = \frac{1}{a}Y_t$, $0 < a < 1$ considered in the literature have either trajectories that tend to $+\infty$ when t tends to infinity, or to zero [see Kamihigashi (2011) for the discussion of the so-called explosive and implosive bubbles]. The process (Y_t) in (2.1) is neither explosive, nor implosive, and has no global trend. Its trajectories feature periodically collapsing bubbles (PCB) as shown in the simulations below. So far, for historical analysis, only two PCB models were used in the economic literature on rational expectations [see e.g. Charemza and Deadman (1995), Taylor and Peel (1998), Psaradakis et al. (2001), Phillips, Wu and Yu (2011)]. The first PCB model is the bubble introduced by Blanchard and Watson (1982). The second PCB model has been introduced by Evans (1991), as an extension of the Blanchard, Watson model. However, its stationarity conditions are unknown and there exists no (quasi) closed-form formula of the term structure of nonlinear predictions for that model.

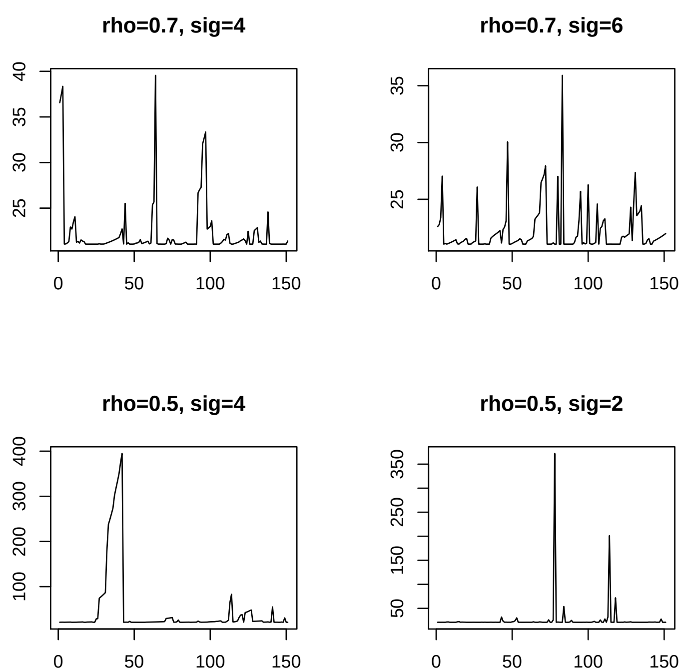
Let us now illustrate the patterns of the trajectories of process (Y_t). In this illustration, the stochastic intensity is defined by $p_t = \Phi(X_t)$, where Φ is the cumulative distribution function of the standard normal and (X_t) is a latent stationary Gaussian autoregressive process such that :

[7] See Appendix 3.3.1 for a discussion of the Blanchard-Watson bubble.

$$X_t = \mu + \rho(X_{t-1} - \mu) + \sigma\sqrt{1-\rho^2}u_t,$$

where the errors $u_t's$ are i.i.d. standard normal, σ is positive and ρ strictly between -1 and 1. In particular, when $\mu = 0$ and $\sigma = 1$, the stationary distribution of X_t is standard normal and p_t is the theoretical Gaussian rank of the value of X_t. Let us choose an initial value $X_0 = \mu$, and set the parameters equal to : $\mu = -1, \eta = 1, a = 0.95$. The two remaining parameters are allowed to vary and take the following four sets of values : $\sigma = 4, \rho = 0.7$; $\sigma = 6, \rho = 0.7, \sigma = 4, \rho = 0.5; \sigma = 2, \rho = 0.5$. By increasing the value of ρ, we increase the persistence[8] of p_t and create more extreme values of p_t in the sense that p_t is approaching 1. This in turn creates an explosive stochastic drift ξ_{2t}. Figure 2 shows the dynamics of (Y_t) with the above four sets of parameter values. It reveals how the parameters determine the frequency of bubbles, the bubble persistence during its growth as well as the bubble's rate of explosion.

Figure 2 : Simulated Paths of Process Y



The values of a and μ are used to generate high values of the stochastic autoregressive coefficient $\frac{1}{a}\frac{Z_t}{p_t}$ in the growth phase of a bubble. Coefficient ρ determines the persistence in the growth phase, whereas the parameter σ has a direct effect on the frequency of bubbles.

[8]This is an intensity clustering effect analogous to the volatility clustering.

2.4 Comparison with the literature on stochastic trees.

2.4.1 Comparison with Cox, Ross, Rubinstein

The model (2.1) can be interpreted as a binomial tree in discrete time that has stochastic branches instead of deterministic branches as in the tree introduced by Cox, Ross and Rubinstein (1979). Let us describe the branching that starts at time t and stems from Y_{t-1}. For illustration, let us assume that (p_t) is i.i.d.. The first branch, $Z_t = 1$, appears with (stochastic) intensity p_t so that the next value of the process on this branch is $Y_t = Y_{t-1}/(ap_t)$. On this branch Y_t follows an autoregression with an explosive (stochastic) autoregressive coefficient ξ_{1t}. Equivalently, by taking the logarithms we find that the returns $\log Y_t - \log Y_{t-1} = \log(ap_t)$ are i.i.d. and positive. The second branch : $Z_t = 0$, is generated with (stochastic) intensity $1 - p_t$ and the process becomes $Y_t = \frac{\eta}{1-a} + \frac{\eta}{a(1-p_t)}$, which is independent of Y_{t-1}. Then, on this branch, the prices are i.i.d. instead of the returns.

The standard interpretation of branches in terms of up/down movements as in the Cox, Ross, Rubinstein tree does not apply directly. The first branch is an up-branch, since $Y_t = Y_{t-1}/(ap_t) > Y_{t-1}$ with a positive stochastic increment. The second branch creates either an up, or a down movement. More precisely, an up movement is observed if and only if $\frac{\eta}{a(1-p_t)} > Y_{t-1} - \frac{\eta}{1-a}$. This arises in two situations : i) when Y_{t-1} is close to $\eta/(1-a)$, which is a reflection of Y_t from its lower bound, and ii) for large Y_{t-1} and p_t close to 1, it accelerates the bubble growth. This dynamics can be related to the "volatility induced mean reversion" when very small (or very large) values of Y_t make the process bounce off the lower bound.

The main differences between the two types of trees are the following:

i) The Cox, Ross, Rubinstein (CRR) tree is designed to create nonstationary (historical and risk-neutral) dynamics for prices, with a suitable submartingale property under the risk-neutral probability. The tree defined in (2.1) can be used in the same way, although its dynamic is stationary.

ii) The CRR tree implies a complete market and a strict relationship between the associated historical and risk-neutral dynamics. The tree defined in (2.1) has stochastic branches, which implies an incomplete market framework, a multiplicity of admissible (discrete time) stochastic discount factors and more flexibility for the coherent choice of historical and risk-neutral parameters [see Appendix 3]. More precisely, if these distributions are chosen in the same semi-parametric family of dynamic models with different parameters [η, a, distribution of (p_t)] for the historical and risk-neutral distributions, then, under the absence of dynamic arbitrage opportunity, these distributions must have the same support in the two worlds. In particular, the parameters a and η have to remain the same in the two worlds. In discrete time, this is the only link between the historical and risk-neutral dynamics induced by the absence of dynamic arbitrage.

As in the CRR tree, the periods of upward movements are endogenous rather than being exogenous functions of time, as assumed in Harvey, Leybourne and Zhu (2020) and Harvey at al (2023)[9]. Thus, we do not want to specify ex-ante the probability of observing bubbles.

2.4.2 Trees as approximations of Continuous Time Models

In the literature, stochastic trees provide tractable approximations of continuous time diffusion models by discretizing both the time and space. Among the examples are the Cox, Ross, Rubinstein binomial tree that approximates the Black, Scholes diffusion [Cox, Ross and Rubinstein (1979)], and the extension considered by Nelson and Ramaswamy (1990) to approximate a more general diffusion $dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t$, which consists in preliminary transforming the process into a diffusion process with constant volatility. The stochastic tree (2.1) is of a different type as it depends on two latent random processes (p_t) and (Z_t), respectively. It is comparable to trees that approximate continuous time stochastic volatility models [see e.g. Gruber and Schweizer (2006), Florescu and Viens (2005), (2008), Akyldirim, Dolinsky and Mete Soner (2014)]. The difference is in the stochastic volatility being replaced by the stochastic intensity. Note that the stochastic tree (2.1) is based on two dependent state variables, which are either (p_t, Z_t), or (ξ_{1t}, ξ_{2t}) [see e.g. Hilliard and Schwartz (1996) for such correlated state variables in the stochastic volatility framework].

To facilitate the comparison with the literature on trees with finite state space conditional on the past, let us study a special case of (Y_t) that arises when the intensity process (p_t) in (2.1) is a Markov chain with two states $\bar{p}_0, \bar{p}_1$, say, and a transition matrix $\begin{pmatrix} \pi_{00} & \pi_{01} \\ \pi_{10} & \pi_{11} \end{pmatrix}$. Then, for lagged intensity $p_{t-1} = \bar{p}_0$, we get a tree with four branches (quadrinomial tree) as in the scheme below :

Scheme : A quadrinomial tree

branch	p_t	Z_t	probability of the branch	future value
$0,1$	$\bar{p}_0$	1	$\pi_{00}\bar{p}_0$	$\eta/(1-a) + (1/a\bar{p}_0)[Y_{t-1} - \eta/(1-a)]$
$0,0$	$\bar{p}_0$	0	$\pi_{00}(1-\bar{p}_0)$	$\eta/(1-a) + \eta/[a(1-\bar{p}_0)]$
$1,1$	$\bar{p}_1$	1	$\pi_{01}\bar{p}_1$	$\eta/(1-a) + (1/a\bar{p}_1)[Y_{t-1} - \eta/(1-a)]$
$1,0$	$\bar{p}_1$	0	$\pi_{01}(1-\bar{p}_1)$	$\eta/(1-a) + \eta/[a(1-\bar{p}_1)]$

[9] Although such models are suitable for detecting the past bubbles, they do not allow for predicting the risk of future bubbles.

For a different value of lagged intensity $p_{t-1} = \bar{p}_1$, the same future values can occur with different probabilities. The lower bound for the trajectories of (Y_t) is : $\eta/(1-a) + \min\{\eta/[a(1-\bar{p}_0)], \eta/[(1-\bar{p}_1)]\}$, which is strictly larger than $\eta/(1-a)$.

3 Nonlinear prediction

This section presents closed-form nonlinear prediction formulas for $Y_{t+H}, H \geq 1$ performed at time t. Next, these term structures of prediction formulas are used to obtain new (quasi) closed-form pricing formulas for European options written on the bubble (asset) price Y_t when the dynamics is considered under a risk-neutral probability. Under a constant (continuously compounded) riskfree rate r, the absence of arbitrage opportunity implies that the price of the underlying asset Y_t satisfies the condition :

$$Y_t = \exp(-r)E(Y_{t+1}|I_t), \tag{3.1}$$

under a risk-neutral probability. This condition is equivalent to the submartingale condition in Proposition 2 with $a = \exp(-r)$, and $r > 0$, as the discount factor.

The pricing formula (3.1) can be compared with the pricing formulas derived from the Hull, White stochastic volatility model [Hull and White (1987), Ball and Roma (1994)], for instance. In a discrete time model, the market is incomplete and the submartingale condition (3.1) provides no information on the historical dynamics of the price process, except for the restriction of common support. In the limiting case when the risk-neutral and historical distributions coincide, the historical dynamics (2.1) features PCB and we price the effects of these PCBs on the derivative payoffs [see Appendix 3 for the link between the historical and risk-neutral dynamics].

3.1 Term structure of nonlinear predictions

The autoregressive representation (2.1) allows us to derive nonlinear predictions of the process at any horizon $H \geq 1$. This is done by means of the conditional moment generating function of the positive process $Y_t - \frac{\eta}{1-a}$, or equivalently by means of the conditional real Laplace transform of the process $\log(Y_t - \frac{\eta}{1-a})$. All these computations are performed conditional on the enlarged information set $I_t = (\underline{Y_t}, \underline{p_t})$, which is the information set of the investor.

Proposition 4 : The conditional Laplace transform is :

$$E\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}\middle| I_t\right]$$

$$= \sum_{h=0}^{H-1}\left\{\frac{\eta^{\alpha}}{a^{\alpha(h+1)}}E\left(\left[(1-p_{t+H-h})\Pi_{k=0}^{h-1}p_{t+H-k}\right]^{1-\alpha}|I_t\right)\right.$$

$$+ \frac{1}{a^{\alpha H}}\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha}E\left[\left(\Pi_{k=0}^{H-1}p_{t+H-k}\right)^{1-\alpha}|I_t\right],$$

where $I_t = (\underline{Y_t}, \underline{p_t})$ and α is any nonnegative scalar such that the conditional expectations:

$E[(1-p_{t+H-h})^{1-\alpha}(\Pi_{k=0}^{h-1}p_{t+H-k})^{1-\alpha}|I_t]$ exist for $h = 0, \ldots, H-1$,

as well as :$E[(\Pi_{k=0}^{H-1}p_{t+H-k})^{1-\alpha}|I_t]$.

Proof : See Appendix 1.4.

Let us introduce the additional assumption:

Assumption A.2: (Y_t) does not Granger cause (p_t), i.e. the conditional distribution of p_t given $\underline{y_{t-1}}, \underline{p_{t-1}}$ depends on $\underline{p_{t-1}}$ only.

Then, under the noncausality assumption A.2, the conditional prediction of $\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}$ is a linear affine function of $\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha}$, with coefficients depending on the current and lagged values of the process (p_t) of stochastic intensities. The prediction formulas are given below for various intensity processes (p_t).

a) i.i.d. stochastic intensities

The nonlinear prediction formula is greatly simplified, when the stochastic intensities $p_t's$ are independent and identically distributed (i.i.d.).

Corollary 2 : Let us assume that the stochastic intensities $p_t's$ are i.i.d.. Then,

$$E\left[\left(Y_{t+H}-\frac{\eta}{1-a}\right)^{\alpha}|I_t\right]$$

$$= c(\alpha)\frac{\eta^{\alpha}}{a^{\alpha}}\frac{1-\dfrac{[\psi(1-\alpha)]^H}{a^{\alpha H}}}{1-\dfrac{\psi(1-\alpha)}{a^{\alpha}}}+\left[\frac{\psi(1-\alpha)}{a^{\alpha}}\right]^H\left(Y_t-\frac{\eta}{1-a}\right)^{\alpha},$$

where :

$$c(\alpha) = E[(1-p_t)^{1-\alpha}], \quad \psi(\alpha) = E(p_t^\alpha),$$

and α is such that $c(\alpha)$ and $\psi(1-\alpha)$ exist[10]

In the framework of i.i.d. stochastic intensities, we get a term structure of nonlinear predictions of the type :

$$E\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t\right] = d_0(H,\alpha) + d_1(H,\alpha)\left(Y_t - \frac{\eta}{1-a}\right)^\alpha, \forall H,$$

where coefficients d_0, d_1 are deterministic functions of term H. Thus, the long run behavior of the term structure of predictions depends on the choice of the common distribution of stochastic intensities (p_t) and on the power coefficient α.

Corollary 3 : For i.i.d. stochastic intensities (p_t), the long run prediction

$\lim_{H\to\infty} E\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t\right]$

exists (and is independent of Y_t), if $\psi(1-\alpha)/a^\alpha < 1$. Otherwise, that limit is infinite.

As mentioned earlier, if $\alpha = 1$, then $\psi(1-\alpha)/a^\alpha = 1/a > 1$ and hence the expectation $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$ is explosive. However, for α sufficiently small, we can expect the marginal expectation $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$ to be finite.

The following examples illustrate these results.

Example 1 : Constant intensities

If $p_t = p$ is constant, we get $c(\alpha) = (1-p)^{1-\alpha}$, and $\psi(1-\alpha) = p^{1-\alpha}$. The condition for the convergence of the series in Corollary 2 is : $p^{1-\alpha}/a^\alpha < 1$, which is equivalent to $\alpha < \log p/\log(ap)$. Since $\log p/\log(ap) = \log p/(\log a + \log p) < 1$, we see that Y_t is non-integrable whereas the conditional expectation of $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$ exists and is finite for any α sufficiently small.

Example 2 : Uniform stochastic intensities

When the stochastic intensities (p_t) follow a uniform distribution on (0,1), we get : $c(\alpha) = E[(1-p_t)^{1-\alpha}] = E(p_t^{1-\alpha}) = \psi(1-\alpha) = \frac{1}{2-\alpha}$. Therefore the conditional expectation

[10]The existence is insured for α between 0 and 1.

of $\left(Y_t - \frac{\eta}{1-\alpha}\right)^\alpha$ exists for $\alpha < 2$. For $\alpha < 2$, the condition for convergence of the long run prediction becomes $\psi(1-\alpha)/a^\alpha = \frac{1}{(2-\alpha)a^\alpha} < 1$.

Example 3 : Stochastic intensities with log-gamma distributions

Let us now assume that $-\log p_t$ follows a gamma distribution $\gamma(\nu)$ with degree of freedom ν. We get :

$$\begin{aligned}
&\psi(1-\alpha) = E(p_t^{1-\alpha}) = E(\exp[(-\log p_t)(\alpha-1)]) \\
&= \int_0^\infty \exp[(\alpha-1)z]\frac{\exp(-z)z^{\nu-1}}{\Gamma(\nu)}dz = \int_0^\infty \exp[-(2-\alpha)z]\frac{z^{\nu-1}}{\Gamma(\nu)}dz = \frac{1}{(2-\alpha)^\nu},
\end{aligned}$$

which exists for $\alpha < 2$.

The condition of convergence of long run predictions becomes $\frac{1}{(2-\alpha)^\nu a^\alpha} < 1$. As a limiting case, this example includes the uniform stochastic intensities of Example 2 for $\nu = 1$.

Example 4 : Beta stochastic intensities

When the intensities (p_t) follow a beta (β, γ) distribution with probability density function (p.d.f.) : $f(p) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta)\Gamma(\gamma)}p^{\beta-1}(1-p)^{\gamma-1}$, with $\beta > 0, \gamma > 0$, we get :

$$\begin{aligned}
\psi(1-\alpha) &= E(p^{1-\alpha}) = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta+\gamma-\alpha+1)}\frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}, \\
\text{and} \quad c(\alpha) &= E[(1-p)^{1-\alpha}] = \frac{\Gamma(\beta+\gamma)}{\Gamma(\beta+\gamma-\alpha+1)}\frac{\Gamma(\gamma-\alpha+1)}{\Gamma(\gamma)}.
\end{aligned}$$

The conditional expectation of $\left(Y_t - \frac{\eta}{1-a}\right)^\alpha$ exists, if $\alpha < \min(\beta,\gamma)+1$.

b) Compound autoregressive log-intensity process

Let us now assume that the (negative) log-intensity process $(\log p_t)$ is a compound autoregressive (CaR) process[11] [Darolles, Gourieroux and Jasiak (2006)]. This process is a Markov process with a conditional Laplace transform which is an exponential affine function of the conditioning value :

[11] The CaR processes are the discrete time analogues of affine processes considered in continuous time [Duffie, Filipovic and Schachermayer (2005)].

$$E[\exp(u \log p_{t+1})|p_t] \equiv \exp[a(u) \log p_t + b(u)], \tag{3.2}$$

where the argument u is nonnegative. Then the affine property of the log-Laplace transform is satisfied at any horizon h and it also holds for the cumulated process, that is,

$$E[\exp u[\log p_{t+1} + \ldots + \log p_{t+h}]|p_t] \equiv \exp[A(h,u) \log p_t + B(h,u)], \tag{3.3}$$

where functions $A(h,u), B(h,u)$ are easily derived recursively. For such a process, the conditional expectation in the prediction formula of Proposition 4 can be simplified. For horizons $h < H$, we get :

$$\begin{aligned} & E[(1-p_{t+H-h})^{1-\alpha}(\Pi_{k=0}^{h-1} p_{t+H-k})^{1-\alpha})|I_t] \\ = \quad & E\{(1-p_{t+H-h})^{1-\alpha} p_{t+H-h}^{A(h,1-\alpha)}|I_t\} \exp B(h, 1-\alpha), \end{aligned}$$

and for $h = H$, we have:

$$E[(\Pi_{k=0}^{H-1} p_{t+H-k})^{1-\alpha}|I_t] = p_t^{A(H,1-\alpha)} \exp B(H, 1-\alpha).$$

The prediction formulas for $h < H$ involve only one-dimensional integrals and are easy to compute. Moreover, the above prediction formula for $h = H$ has a closed form.

Example 5 : Autoregressive Gamma process

This process is the time discretized Cox, Ingersoll, Ross process [Cox, Ingersoll and Ross (1985)]. The short-term conditional Laplace transform is [Gourieroux and Jasiak (2006)] :

$$E[\exp(u \log p_{t+1})|I_t] = \exp\left[\frac{u}{1+u} \log p_t - \delta \log(1+u)\right],$$

where δ is a parameter, $a(u) = \dfrac{u}{1+u}$, and $b(u) = -\delta \log(1+u)$. It provides us a closed-form prediction formula.

3.2 European Option pricing

Let us now consider European options written on a bubble (asset) price $Y_t, Y_t > 0$. Under the absence of dynamic arbitrage opportunity and the assumption of constant (continuously compounded) riskfree rate r, $r > 0$, there exists a risk-neutral distribution Q such that the value at date t of the future payoff $g(Y_{t+1})$ is given by $\exp(-r)E^Q(g(Y_{t+1}|Y_t))$, $\forall g$. In particular,

$$E^Q(Y_{t+1}|I_t) = \exp(r)Y_t. \tag{3.4}$$

This is a submartingale condition. Then, we can consider the derivative pricing formula when, under the risk-neutral dynamic, the price process (Y_t) is stationary and satisfies (2.1) with $a = exp(-r)$. For ease of exposition we do not change the notation so that the parameters remain without the risk-neutral superscript (see Appendix 3.2). The price at date t of a European derivative paying $\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha$ at date $t+H$ is equal to :

$$\exp(-rH)E^Q\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t\right] = a^H E^Q\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t\right],$$

and the pricing formula follows directly from Proposition 4. The risk-neutral dynamics of (Y_t) cannot be chosen arbitrarily, as the current price of the underlying asset needs to exist and be finite. For instance, in the case of $\alpha = 1$, the price has to exist (see the remark after Corollary 3). When the conditional expectation does not exist, some derivative payoffs cannot be traded on the derivative market, because they are too expensive and/or too costly to hedge. Therefore they are non-insurable. This is especially important given that we are interested in the pricing/hedging of rare and extreme spikes associated with periodically collapsing speculative bubbles. As another example, let us consider quadratic payoffs for $\alpha = 2$. In such a case, the models in Examples 2-3 cannot be used for pricing these quadratic payoffs, whereas models in Examples 1 and 4 can, with $\min(\beta, \gamma) > 1$ in the latter case.

Example 6 : Quadratic Payoff

Let us consider again i.i.d. stochastic probabilities with finite conditional expectation for $\alpha = 2$ (see Example 4). From Corollary 2 and the submartingale condition, it follows that the price of a historical volatility[12] paying Y_{t+H}^2 at date $t+H$ is :

$$V(Y_t, H) \equiv 2\frac{\eta}{1-a}Y_t - \frac{\eta^2 a^H}{(1-a)^2} + c(2)\eta^2 a^{H-2}\frac{1-\frac{\psi(-1)^H}{a^{2H}}}{1-\frac{\psi(-1)}{a^2}} + \frac{\psi(-1)^H}{a^H}\left(Y_t - \frac{\eta}{1-a}\right)^2. \tag{3.5}$$

This formula can be compared with the price of the quadratic payoff derived from the Black-Scholes formula with volatility parameter σ and continuous compounding parameter $a = \exp(-r)$. We get :

$$V^{BS}(Y_t, H) = \frac{\exp(\sigma^2 H)}{a^H}Y_t^2. \tag{3.6}$$

[12]Such historical volatility is the key component of a variance swap.

The proposed formula (3.5) for $V(t,H)$ appears more complex and also more flexible than the Black-Scholes formula, as it involves a "scale" parameter η, and two stochastic intensity parameters: parameter $\psi(-1)$, which is larger or equal to 1, and parameter c(2), which is also larger or equal to 1. Moreover it depends on the past and current values through an affine combination of Y_t and Y_t^2 instead of Y_t^2 only as in $V^{BS}(Y_t, H)$.

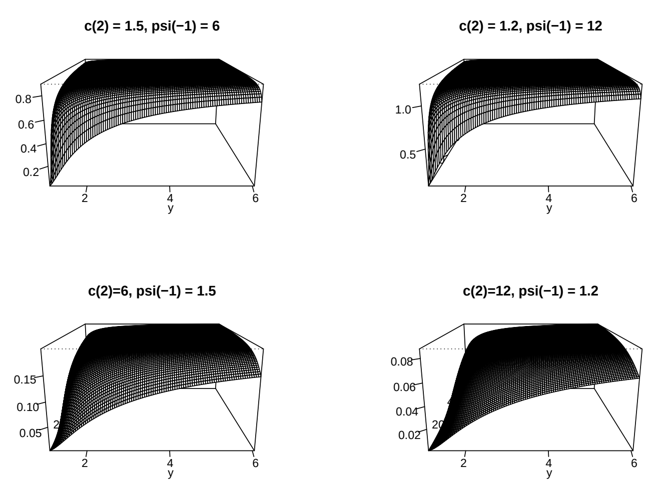
For a large current value of Y_t, the two formulas are equivalent when $\psi(-1) = \exp(\sigma^2)$. Thus, the term structures of prices of quadratic payoffs are equivalent at those infrequent times when the bubble takes extreme values (see Figure 2). Otherwise, the formulas are necessarily different, because the Black-Scholes model does not account for the possibility of extreme bubbles.

By analogy with the standard practice of computing the Black-Scholes implied volatility, we can compute the quantity :

$$\sigma^{BS}(Y_t, H) = \sqrt{\frac{1}{H} \log \left[\frac{V(Y_t, H) a^H}{Y_t^2} \right]}, \tag{3.7}$$

obtained by inverting the Black-Scholes formula for historical variance (3.6) with the value of bubble derivative substituted for $V^{BS}(Y_t, H)$. Figure 3 provides the Black-Scholes implied volatility surfaces for H and $Y_t = y$ varying, and the parameters $\eta, a, c(2), \psi(-1)$ held fixed at constant values.

Figure 3 : Implied BS Volatility Surfaces



The parameters $c(2) = E[1/(1-p)]$ and $\psi(-1) = E(1/p)$ are constrained by the inequality given below. For instance, since $\frac{1}{1-p} = 1 + \frac{1}{1/p - 1}$, it follows from the Jensen's inequality that :

$$c(2) = E[1/(1-p)] = 1 + E[1/(\frac{1}{p} - 1)] \geq 1 + \frac{1}{E(1/p) - 1} = \frac{\psi(-1)}{\psi(-1) - 1}.$$

Similarly, we could derive an upper bound for $c(2)$ for a given $\psi(-1)$. To satisfy such inequality constraints, we consider beta stochastic intensities (see Example 4). We get :

$$c(2) = (\beta + \gamma - 1)/(\gamma - 1), \psi(-1) = (\beta + \gamma - 1)/(\beta - 1),$$

where β and γ are larger than 1 to satisfy the condition of existence of the conditional expectation.

The values of the parameters take these constraints into account. The values of $\eta = 0.01$ and $a = 0.99$ imply a lower bound of 1 for process (Y_t). The selected pairs for $[c(2), \psi(-1)]$ (or equivalently for (β, γ)) are :

$$\begin{array}{lll} \beta = 2,\ \gamma = 5 & \text{yielding} & c(2) = 1.5, \psi(-1) = 6; \\ \beta = 2,\ \gamma = 11 & \text{yielding} & c(2) = 1.2, \psi(-1) = 12; \\ \beta = 5,\ \gamma = 2 & \text{yielding} & c(2) = 6, \psi(-1) = 1.5; \\ \beta = 11, \gamma = 5 & \text{yielding} & c(2) = 12, \psi(-1) = 1.2. \end{array}$$

The two first (resp. second) scenarios correspond to infrequent bubbles with high rates of explosion (resp. frequent bubbles with small rates of explosion).

Figure 3 shows that for small and medium values y, the price of volatility swap is higher than the limiting Black-Scholes price. Thus, the model adjusts for the mispricing that occurs in practice on option markets. As noted in Coval and Shumway (2001), Bakshi and Kapadia (2003), the current market is "generating surprising large returns from selling crash insurance via out-of-the money put options".

3.3 Continuous time analogue

The literature on derivative pricing examines the relationship between the discrete and continuous time models to find out if a sequence of discrete time models at increasing frequencies tends to a well-defined continuous time model [see e.g. Stroock and Varadhan (1979)]. For a Markov process (Y_t), this can be done by considering the infinitesimal generator : $Gf(y) = \lim_{dt \to 0} \frac{1}{dt} E[f(Y_{t+dt}) - f(Y_t) | Y_t = y]$. Since the intensity process has exogenous dynamics, the infinitesimal generator is derived conditional on the intensity process (p_t).

In Appendix 2 we find that the limiting continuous time dynamics of (Y_t) can be written as :

$$
\begin{aligned}
dY_t &= (\alpha + \pi_t)(Y_t - \frac{\eta}{\alpha})dt + [\eta(\frac{1}{\alpha} + \frac{1}{\pi_t}) - Y_t]dN_t && (3.8)\\
&= \alpha Y_t dt + [\eta(\frac{1}{\alpha} + \frac{1}{\pi_t}) - Y_t](dN_t - \pi_t dt), && (3.9)
\end{aligned}
$$

where (N_t) is a counting process with stochastic intensity $\pi_t, \pi_t > 0$. Parameters α, η and process (π_t) are the infinitesimal analogues of parameters a, η and of the process of probabilities (p_t). Thus, conditional on the exogenous process of stochastic intensities, (Y_t) is a jump process with a predetermined drift and jump magnitude, both being functions of y_t and π_t. Model (3.8)-(3.9) needs to be completed by a specification for the exogenous dynamics of stochastic intensity (π_t). For example, when (π_t) follows a Cox, Ingersoll, Ross diffusion, a continuous time analogue of Example 5 is obtained :

$$
d\pi_t = (b - c\pi_t)dt + \sqrt{\pi_t}\sigma dW_t, \tag{3.10}
$$

where (W_t) is a Brownian motion, and parameters are such that $b > 0, c > 0, \sigma > 0$, and $2b/\sigma^2 > 1$ to ensure the existence and stationarity of process (π_t).

The limiting jump dynamics of process (Y_t) is easy to anticipate. Note, however that model (3.8)-(3.9) differs from the model introduced in Bates (2008) for crash risk. More specifically, the model of Bates (2008) is :

$$
dY_t \equiv \mu dt + \sigma dW_t + \gamma dN_t, \tag{3.11}
$$

where (W_t) is a Brownian motion, while (N_t) is a Poisson process with constant intensity λ, and μ, σ, γ are constant parameters. Like the Bates' model, model (3.8)-(3.9) is a two-factor model. It depends on more parameters, however. More specifically, the Bates' model has 3 parameters, while model (3.8)-(3.9) has 5 parameters. As mentioned earlier in the text, these additional parameters determine the frequency of the bubbles and the persistence of bubble during its growth episode.

The continuous time model (3.8)-(3.9) inherits some properties of the stochastic tree (2.1), such as the local drift, which is linear in Y_t, and the positive submartingale property.

4 Concluding Remarks

This paper introduces a new tree model for the dynamics of price processes with recurrent bubbles and a new approach for pricing derivatives written on bubble assets. This tree accounts for both the stationarity and submartingale property of such bubble assets, both under the historical and risk-neutral probabilities. The tree representation of the new model makes it comparable to other trees for derivative pricing that already exist in the literature. The model provides (quasi) closed-form pricing formulas that arise as alternatives to the Black-Scholes formulas. Therefore, the proposed specification is a relevant addition to bubble models that include noncausal stationary processes [see, e.g. Gourieroux and Lu (2023) for bubble asset pricing based on noncausal processes], as well as the double autoregressive model [Ling (2004)], the model with volatility induced mean reversion [Conley, Hansen, Luttmer, Scheinkman (1997)] and the models with co-jumps [Hou et al. (2020)]. These specifications can be easily extended to multivariate processes and the analysis and pricing of common bubbles [Swensen (2022), Cubadda et. al (2019), (2023), Evripidou et al. (2022), Hall, Jasiak (2024), Gourieroux and Jasiak (2024)].

For further research, there remains a practical question: can we disentangle the local trends of stationary bubbles from the global trends in nonstationary processes with random walk components? Regardless, there is a need for pricing formulas that accommodate both types of trend with different consequences for pricing. More generally in a multivariate model $X_t = X_{f,t} + \beta Y_t$ the univariate component Y_t with the proposed dynamics (2.1) can be introduced to represent a systemic bubble factor. More precisely, let us assume that $X_{f,t}$ is independent of Y_t, and such that $\log X_{f,t}$ follows a Gaussian random walk with drift under the risk-neutral distribution. Then, by using the Laplace transforms, it is easy to deduce from the pricing formulas of $X_{f,t}$ and Y_t, a closed-form pricing formula for the derivatives written on X_t. This formula will depend on the parameters involved in the Black-Scholes and bubble pricing formulas. If prices of derivatives written on X_t are available at date t, then the (risk-neutral) parameters can be estimated as it is commonly done by a calibration approach (i.e. a method of moments) applied cross-sectionally. This would provide a series of estimates indexed by time that can be used to test the risk-neutral specification.

References

[1] Akharif, A. and M., Hallin (2003): "Efficient Detection of Random Coefficients in Autoregressive Models", The Annals of Statistics, 31, 675-704.

[2] Akyildirim, E., Dolinsky, Y., and M., Mete Soner (2014) : "Approximating Stochastic Volatility by Recombinant Trees", Ann. Appl. Probab., 24, 2176-2205.

[3] Bakshi, G., and N., Kapadia (2003) : "Delta-Hedged Gains and the Negative Market Volatility Risk Premium", Review of Financial Studies, 16, 527-566.

[4] Ball, C., and A., Roma (1994) : "Stochastic Volatility Option Pricing", Journal of Financial and Quantitative Analysis, 29, 589-607.

[5] Banerjee, A., Chevillon, G., and M., Kratz (2020): "Probabilistic Forecasting of Bubbles and Flash Crashes", The Econometrics Journal, 23, 297-315.

[6] Bates, D. (2008) : "The Market for Crash Risk", Journal of Economic Dynamics and Control, 32, 2291-2321.

[7] Bec, F., Nielsen, H. B. and S. Saidi : "Mixed Causal-Noncausal Autoregressions: Bimodality Issues in Estimation and Unit Root Testing ", Oxford Bulletin of Economics and Statistics, 82, 1413-1428.

[8] Berkes, I., Horvath, L., and S., Ling (2003): "Estimation in Nonstationary Random Coefficient Autoregressive Models", Journal of Time Series Analysis, 30, 395-416.

[9] Blanchard, O., and M., Watson (1982) : "Bubbles, Rational Expectations and Financial Markets", in Crises in the Economic and Financial Structure, ed. by P., Wachtel, 295-316, Lexington MA / DC Heathand Company.

[10] Brandt, A. (1986) : "The Stochastic Equation $Y_{n+1} = A_n Y_n + B_n$ with Stationary Coefficients", Advances in Applied Probability, 8, 211-220.

[11] Cavaliere, G., Nielsen, H. B, and A. Rahbek (2020): "Bootstrapping Noncausal Autoregressions with Application to Explosive Bubble Modelling", Journal of Business and Economic Statistics, 38, 55-67.

[12] Charemza, W., and D., Deadman (1995) : "Speculative Bubbles with Stochastic Explosive Roots : The Failure of Unit Root Testing", Journal of Empirical Finance, 2, 153-163.

[13] Conley, T., Hansen, L., Luttmer, E. and J. Scheinkman (1997): "Short Term Interest Rates as Subordinated Diffusions", Review of Financial Studies, 10, 525-577.

[14] Conlisk, J. (1974): "Stability in a Random Coefficient Model", International Economic Review, 15, 529-533.

[15] Conlisk, J. (1976): "A Further Note on Stability in a Random Coefficient Model", International Economic Review, 17, 759-764.

[16] Coval, J., and T., Shumway (2001) : "Expected Option Returns", Journal of Finance, 56, 983-1009.

[17] Cox, J., Ingersoll, J., and S., Ross (1985) : "A Theory of the Term Structure of Interest Rates", Econometrica, 53, 385-408.

[18] Cox, J., Ross, S., and M., Rubinstein (1979) : "Option Pricing : A Simplified Approach", Journal of Financial Economics, 7, 229-263.

[19] Cubadda, G., Hecq, A. and S. Telg (2019): "Detecting Comovements in Non-Causal Time Series", Oxford Bulletin of Economics and Statistics, 81, 697-715.

[20] Cubadda, G., Hecq, A. and E. Voisin (2023): "Detecting Common Bubbles in Multivariate Mixed Causal-Noncausal Models", Econometrics, 11(1), 9; https://doi.org/10.3390/econometrics11010009.

[21] Darolles, S., Gourieroux, C., and J., Jasiak (2006) : "Structural Laplace Transform and Compound Autogressive Models", Journal of Time Series Analysis, 27, 477-503.

[22] Duffie, D., Filipovic, D., and W., Schachermayer (2005) : "Affine Processes and Applications in Finance", Annals of Applied Probability, 13, 984-1053.

[23] Evans, G. (1991) : "Pitfalls in Testing for Explosive Bubbles in Asset Prices", The American Economic Review, 81, 922-930.

[24] Evripidou, A., Harvey, D., Leybourne, S., and R., Sollis (2022): "Testing for Co-Explosive Behaviour in Financial Time Series", Oxford Bulletin of Economics and Statistics 84, 624-650.

[25] Florescu, I., and F., Viens (2005) : "A Binomial Tree Approach to Stochastic Volatility Driven Model of the Stock Price", Annals of the University of Craiova, 32, 126-142.

[26] Florescu, I., and F., Viens (2008) : "Stochastic Volatility : Option Pricing Using a Multinomial Recombining Tree", Applied Mathematical Finance, 15, 151-181.

[27] Fouque, J-P., Papanicolaou, G., and K., Sircar (2000) : "Derivatives in Financial Markets with Stochastic Volatility", Cambridge University Press.

[28] Gourieroux, C., and A., Hencic (2015) : "Noncausal Autoregressive Model in Application to Bitcoin/USD Exchange Rates", in Huynh V., Kreinovich, V., Sriboonchitta, S., and K., Suriya, eds., Econometrics of Risk, Studies in Computational Intelligence, 583, 17-40.

[29] Gourieroux, C., Hencic, A. and J. Jasiak (2021) : "Forecast Performance and Bubble Analysis in Noncausal MAR(1,1) Processes", Journal of Forecasting, 40, 301-326.

[30] Gourieroux, C., and J., Jasiak (2006) : "Autoregressive Gamma Processes", Journal of Forecasting, 25, 129-152.

[31] Gourieroux, C., and J., Jasiak (2016) : "Filtering, Prediction and Simulation Methods for Noncausal Processes", Journal of Time Series Analysis, 37, 405-430.

[32] Gourieroux, C., and J., Jasiak (2017) : "Noncausal Vector Autoregressive Process: Representation, Identification and Semi-Parametric Estimation", Journal of Econometrics, 200, 118-134.

[33] Gourieroux, C., and J., Jasiak (2019) : "Robust Tests of the Martingale Hypothesis", Econometrics and Statistics, 9, 17-41.

[34] Gourieroux, C., and J., Jasiak (2024) : "Nonlinear Fore(Back)cast and Innovation Filtering for Causal-Noncausal VAR Models", ArXiv 2205.09922

[35] Gourieroux, C., Jasiak, J., and A., Monfort (2020) : "Stationary Bubble Equilibria in Rational Expectation Models", Journal of Econometrics, 218, 714-735.

[36] Gourieroux, C., and Y., Lu (2023): "Noncausal Affine Processes with Application to Derivative Pricing", Mathematical Finance, 33, 766-796.

[37] Gourieroux, C., and J.M., Zakoian (2017) : "Local Explosion Bubble Modelling by Noncausal Process", Journal of the Royal Statistical Society, B, 79, 737-756.

[38] Gruber, U., and M., Schweizer (2016) : "A Diffusion Limit for Generalized Correlated Random Walks", J. App. Probab., 43, 60-73.

[39] Hall, M. and J., Jasiak (2024): "Modelling Common Bubbles in Cryptocurrency Prices", working paper at www.jjstats.com

[40] Harvey, D., Leybourne, S., Sollis R., and R., Taylor (2016): "Tests for Explosive Financial Bubbles in the Presence of Nonstationary Volatility", Journal of Empirical Finance, 38, 548-574.

[41] Harvey, D., Leybourne, S., Tatlow, B., and Y., Zhu (2023): "Unit Root Tests for Explosive Financial Bubbles in the Presence of Deterministic Level Shifts", DP, University of Nottingham.

[42] Harvey, D., Leybourne, S., and Y., Zhu (2020): "Sign Based Unit Root Tests for Explosive Financial Bubbles in the Presence of Deterministic Time-Varying Volatility", Econometric Theory, 36, 122-169.

[43] Hecq, A., and E., Voisin (2021): "Forecasting Bubbles with Mixed Causal-Noncausal Autoregressive Models", Econometrics and Statistics, 20, 29-45.

[44] Hecq, A., and E., Voisin (2023): "Predicting Crashes in Oil Prices During the Covid-19 Pandemic with Mixed Causal-Noncausal Models", Advances in Econometrics, in Essays in Honor of Joon Park, Econometric Methodology in Empirical Applications, 45, 209-233.

[45] Hilliard, J., and A., Schwartz (1996) : "Binomial Option Pricing Under Stochastic Volatility and Correlated State Variables", The Journal of Derivatives, 4, 23-39.

[46] Horvath, L., and L., Trapani (2019): "Testing for Randomness in a Random Coefficient Autoregressive Model", Journal of Econometrics, 209, 338-352.

[47] Hou, A., Wang, W., Chen, C., and W., Hardle (2020): "Pricing Cryptocurrency Options", Journal of Financial Econometrics, 18, 250-279.

[48] Hull, J., and A., White (1987) : "The Pricing of Options on Assets with Stochastic Volatilities", Journal of Finance, 42, 281-300.

[49] Hwang, S., and I., Basawa (1998): "Parameter Estimation for Generalized Random Coefficient Autoregressive Processes", Journal of Statistical Planning and Inference, 68, 323-337.

[50] Jarrow, R. Protter, P. and K. Shimbo (2006): "Asset Price Bubbles in a Complete Market", Adv. Math. Finance in honor of D. Madan, 105-130.

[51] Jarrow, R. Protter, P. and K. Shimbo (2010): "Asset Price Bubbles in Incomplete Markets", Mathematical Finance, 20, 145-185.

[52] Johansen, S., and T., Lange (2011) : "Some Econometric Results for the Blanchard-Watson Bubble Model", CREATES DP.

[53] Kamihigashi, T. (2011) : "Recurrent Bubbles", Japanese Economic Review, 62, 27-62.

[54] Lee, J., and P., Phillips (2016) : "Asset Pricing with Financial Bubble Risk", Journal of Empirical Finance, 38, 590-622.

[55] Leisen, D. (2000) : "Stock Evolution Under Stochastic Volatility : A Discrete Approach", Journal of Derivatives, 8, 9-27.

[56] Leybourne, S., McCabe, P., and T., Mills (1996): "Randomized Unit Root Processes for Modelling and Forecasting Financial Time Series: Theory and Applications", Journal of Forecasting, 15, 253-270.

[57] Ling, S. (2004): "Estimation and Testing Stationarity for Double Autoregressive Models", Journal of the Royal Statistical Society, Series B, 66, 63-78.

[58] Nelson, D., and K., Ramaswamy (1990) : "Simple Binomial Process as Diffusion Approximation in Financial Models", The Review of Financial Studies, 3, 393-430.

[59] Nicholls, D., and B., Quinn (1981a): "The Estimation of Random Coefficient Autoregressive Models II", Journal of Time Series Analysis, 2, 185-203.

[60] Nicholls, D., and B., Quinn (1981b): "Multiple Autoregressive Models with Random Coefficients", Journal of Multivariate Analysis, 11, 185-198.

[61] Nicholls, D., and B., Quinn (1981b): "Random Coefficient Autoregressive Models: an Introduction", New-York, Springer Verlag.

[62] Phillips, P., Wu, Y., and J., Yu (2011) : "Explosive Behavior in the 1990s Nasdaq : When Did Exuberance Escalate Asset Values", International Economic Review, 52, 201-226.

[63] Phillips, P., Shi, S., and J., Yu (2015) : "Testing for Multiple Bubbles: Historical Episodes of Exuberance and Collapse in S&P500", International Economic Review, 56, 1043-1078.

[64] Phillips, P., and J., Yu (2011) : "Dating the Timeline of Financial Bubbles During the Subprime Crisis", Quantitative Economics, 2, 455-491.

[65] Psaradakis, Z., Sola, M., and F., Spagnolo (2001) : "A Simple Procedure for Detecting Periodically Collapsing Rational Bubbles", Econometrics Letters, 72, 317-323.

[66] Regis, M., Serra, P. and E., Van den Heuvel (2022): "Random Autoregressive Models: A Structural Overview", Econometric Reviews, 41, 207-230.

[67] Stroock, D., and S., Varadhan (1979) : "Multidimensional Diffusion Processes", Berlin, Springer Verlag.

[68] Swensen, A. (2022): "On Causal and Noncausal Cointegrated Vector Autoregressive Time Series", Journal of Time Series Analysis, 42, 178-196.

[69] Taylor, M., and D., Peel (1998) : "Periodically Collapsing Stock Price Bubbles : A Robust Test", Economics Letters, 61, 221-228.

[70] Tirole, J. (1985): "Asset Bubbles and Overlapping Generations", Econometrica, 53, 1499-1528.

Appendix 1

1.1 Proof of Proposition 1

i) Under Assumption A.1, we have :

$P[\Pi_{k=0}^{H-1} Z_{t-k} = 1 | (p_t)] = \Pi_{k=0}^{H-1} p_{t-k}$, which tends a.s to zero.

Since $\Pi_{k=0}^{H-1} Z_{t-k}$ takes only values 0 and 1, it follows that $\Pi_{k=0}^{H-1} Z_{t-k}$ is decreasing and tends a.s. to zero. Moreover, as (Y_t) is stationary, the last term in (2.2) tends a.s. to zero as well, when H tends to infinity. Therefore, a strictly stationary solution of (2.1), if it exists, has the nonlinear moving average (MA) representation (2.3). In particular, it is unique.

ii) The nonlinear MA (∞) representation (2.3) implies that, conditional on any given path of $(p_t), Y_t$ follows a discrete distribution and the values of Y_t :

$$\frac{\eta}{1-a} + \frac{\eta}{a} \frac{1}{a^h} \frac{1}{(1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k}},$$

have probabilities $(1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k}, h = 0, \ldots, \infty$. Indeed, under Assumption A.1, we can define the sum of these probabilities :

$\sum_{h=0}^{\infty} ((1-p_{t-h}) \Pi_{k=0}^{h-1} p_{t-k})$, which is equal to 1. Therefore, the conditional distribution of Y_t given I_{t-1} exists and is independent of time t, which implies a homogeneous transition. Since the process (p_t) is strongly stationary and this transition is homogeneous, we deduce that the process (Y_t) itself is strictly stationary.

1.2 Proof of Proposition 2

From (2.1) we get :

$$\begin{aligned}
E(Y_t | \underline{Y_{t-1}}, \underline{p_t}) &= \frac{\eta}{1-a} + \frac{1}{a p_t} (Y_{t-1} - \frac{\eta}{1-a}) E(Z_t | \underline{Y_{t-1}}, \underline{p_t}) + \frac{\eta}{a(1-p_t)} [1 - E(Z_t | \underline{Y_{t-1}}, \underline{p_t})] \\
&= \frac{\eta}{1-a} + \frac{1}{a} (Y_{t-1} - \frac{\eta}{1-a}) + \frac{\eta}{a} \\
&= \frac{1}{a} Y_{t-1} + \frac{\eta}{1-a} - \frac{1}{a} \frac{\eta}{1-a} + \frac{\eta}{a} \\
&= \frac{1}{a} Y_{t-1}.
\end{aligned}$$

1.3 Proof of Proposition 3

We have :

$$\begin{aligned} EY_t &= EE[Y_t|p_t, p_{t-1}, \ldots] = E\left(\frac{\eta}{1-a} + \frac{\eta}{a}\sum_{h=0}^{\infty}\frac{1}{a^h}\right) \text{ (by (2.3))} \\ &= +\infty, \end{aligned}$$

since the geometric series $\sum_{k=0}^{\infty}\frac{1}{a^h}$ diverges for $a \in (0,1)$.

1.4 Proof of Proposition 4

It follows from (2.2) that :

$$\begin{aligned} (Y_{t+H} - \frac{\eta}{1-a})^\alpha &= \frac{\eta^\alpha}{a^\alpha}\sum_{h=0}^{H-1}\Big\{\frac{1}{a^{\alpha h}}\frac{1}{[(1-p_{t+H-h})\Pi_{k=0}^{h-1}p_{t+H-k}]^\alpha}(1-Z_{t+H-h}) \\ &\quad \Pi_{k=0}^{h-1}(1-Z_{t+H-k})\Big\} \\ &+ \frac{1}{a^{\alpha H}}\frac{1}{(\Pi_{k=0}^{H-1}p_{t+H-k})^\alpha}(\Pi_{k=0}^{H-1}Z_{t+H-k})(Y_t - \frac{\eta}{1-a})^\alpha, \end{aligned}$$

by using the interpretation of the $Z_t's$ as indicator variables that implies $Z_t^\alpha = Z_t, (1-Z_t)^\alpha = 1-Z_t$.

The result follows by noting that :

$$\begin{aligned} & E_t\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha\right] \\ =& E\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t\right] \\ =& E\left\{E\left[\left(Y_{t+H} - \frac{\eta}{1-a}\right)^\alpha |I_t, \underline{p_{t+H}}\right]|I_t\right\}, \text{ by iterated expectations.} \end{aligned}$$

Appendix 2

Infinitesimal Generator

Let us consider the behavior of process (Y_t) at a small time step δ. The recursive equation (2.1), with δ-indexed parameters, becomes :

$$Y_{t+\delta} = \frac{\eta_\delta}{1-a_\delta} + \frac{1}{a_\delta p_{\delta t}}(Y_t - \frac{\eta_\delta}{1-a_\delta})Z_{\delta t} + \frac{\eta_\delta}{a_\delta(1-p_{\delta t})}(1-Z_{\delta t}), \tag{a.1}$$

where $Z_{\delta t} \sim \mathcal{B}(1, p_{\delta t})$ and t is a multiple of δ. We assume that parameters and intensity process depend on δ as follows :

$$\eta_\delta = \eta\delta, a_\delta = \exp(-\alpha\delta), p_{\delta t} = \exp(-\pi_t\delta), \tag{a.2}$$

where $\eta > 0, \alpha > 0$, and (π_t) is a positive process. π_t can be interpreted as the infinitesimal intensity on the branch without an autoregressive effect of Y_{t-1}. Then we have :

$$\begin{aligned} \frac{\eta_\delta}{1-a_\delta} &\approx \frac{\eta\delta}{\alpha\delta - \alpha^2\frac{\delta^2}{2}} = \frac{\eta}{\alpha(1-\frac{\alpha}{2}\delta)} = \frac{\eta}{\alpha}(1+\frac{\alpha}{2}\delta) = \frac{\eta}{\alpha} + \frac{\eta}{2}\delta, \\ \frac{1}{a_\delta} &\approx 1+\alpha\delta, \frac{\eta_\delta}{a_\delta} \sim \eta\delta. \end{aligned}$$

Therefore equation (a.1) becomes :

$$Y_{t+\delta} \approx \frac{\eta}{\alpha} + \frac{\eta\delta}{2} + (1+\alpha\delta)[Y_t - (\frac{\eta}{\alpha} + \frac{\eta\delta}{2})]\frac{Z_{\delta t}}{p_{\delta t}} + \eta\delta\frac{1-Z_{\delta t}}{1-p_{\delta t}}, \tag{a.3}$$

where $Z_{\delta t}$ is Bernoulli distributed $B(1, p_{\delta t})$.

Because the process is nonnegative, its transition conditional on π_t is characterized by its conditional cumulant generating function given below and written for a nonnegative argument u :

$$
\begin{aligned}
\psi_{t,\delta}(u) &= E[\exp(-uY_{t+\delta})|Y_t,\pi_t] \\
&\approx \exp[-u(\frac{\eta}{\alpha}+\frac{\eta}{2}\delta)]\{p_{\delta t}\exp\{-u\frac{[1+\alpha\delta]}{p_{\delta t}}[Y_t-(\frac{\eta}{\alpha}+\frac{\eta}{2}\delta)]\} \\
&+ (1-p_{\delta t})\exp(-\frac{u\eta\delta}{1-p_{\delta t}})\} \\
&\approx \exp(-u\eta/\alpha)\exp(-\frac{u\eta\delta}{2})[(1-\pi_t\delta)\exp\{-u[1+(\alpha+\pi_t)\delta](Y_t-\frac{\eta}{\alpha}-\frac{\eta}{2}\delta)] \\
&+ \pi_t\delta\exp(\frac{-u\eta}{\pi_t})\} \\
&\approx \exp(-uY_t)(1-\frac{u\eta\delta}{2})(1-\pi_t\delta)[1-u\delta(\alpha+\pi_t)(Y_t-\frac{\eta}{\alpha})+\frac{u\eta\delta}{2}] \\
&+ \pi_t\delta\exp[-u\eta(\frac{1}{\alpha}+\frac{1}{\pi_t})] \\
&\approx \exp(-uY_t)[1+\delta(-\pi_t-u(\alpha+\pi_t)(Y_t-\frac{\eta}{\alpha})]+\pi_t\delta\exp[-u\eta(\frac{1}{\alpha}+\frac{1}{\pi_t})].
\end{aligned}
$$

It follows that :

$$
\begin{aligned}
&\frac{1}{\delta}[\psi_{t,\delta}(u)-\psi_{t,0}(u)] \\
=\ &\frac{1}{\delta}E[\exp(-uY_{t+\delta})-\exp(-uY_t)|Y_t,\pi_t] \\
\approx\ &-\exp(-uY_t)[\pi_t+u[(\alpha+\pi_t)(Y_t-\frac{\eta}{\alpha})]]+\exp[-u\eta(\frac{1}{\alpha}+\frac{1}{\pi_t})]\pi_t.
\end{aligned} \tag{a.4}
$$

This formula provides the differential form of the infinitesimal generator for exponential transformations $f(y)=\exp(-uy), \forall u$. It also holds for any differentiable function f derived from these exponential transformations by linear combination and closure.

Thus, the necessary form of the infinitesimal generator of the Markov process (Y_t) conditional on process (π_t) is :

$$
Gf(y)=\pi[-f(y)+f(\eta(\frac{1}{\alpha}+\frac{1}{\pi}))]+\frac{df(y)}{dy}[(\alpha+\pi)(y-\frac{\eta}{\alpha})].
$$

This form is interpreted in terms of jump processes in Section 3.3.

Appendix 3

Link Between Historical and Risk-Neutral Dynamics

3.1 Absence of Arbitrage Opportunity AAO)

Let us briefly review the AAO condition for an asset with infinite lifetime and price Y_t. We assume that the available information set is $I_t = (\underline{Y_t}, \underline{p_t})$. We have to distinguish the conditioning operator (or historical distribution) characterized by the transition density $\pi(y_t, p_t|y_{t-1}, p_{t-1})$ (assuming a Markov process) and the pricing operator that is constructed from a stochastic discount factor (sdf) denoted by $M_{t,t+1}$, which is a function of I_{t+1} such that:

$$Y_t = E(Y_{t+1}M_{t,t+1}|I_t),$$

for instance. This pricing formula can be rewritten as:

$$Y_t = E(M_{t,t+1}|I_t)E(Y_{t+1}\frac{M_{t,t+1}}{E_t M_{t,t+1}}|I_t) = \exp(-r)E^Q(Y_{t+1}|I_t)$$

where r is the risk-free rate and Q denotes the risk-neutral probability with the conditional transitions $q(y_t, p_t|y_{t-1}, p_{t-1})$ such that

$$\frac{M_{t,t+1}}{E_t M_{t,t+1}} = \frac{q(y_t, p_t|y_{t-1}, p_{t-1})}{\pi(y_t, p_t|y_{t-1}, p_{t-1})}.$$

Let us now examine if one can find independently the conditioning operator and the pricing operator (or, equivalently the risk-neutral probability):

Proposition A.1 i) In the discrete time framework, the dynamic AAO condition is the existence and the strict positivity of the sdf, or equivalently of the change of probability.

ii) Moreover, in general there exists an infinite number of pricing operators associated with a given conditioning operator.

3.1 Compatibility of historical and risk-neutral stochastic trees

In joint modelling of historical and risk-neutral dynamics, it is common to choose them in the same family of distributions. In our framework, we can consider the family defined by (2.1) and parametrized by a, η and the transition probability $g(p_t|p_{t-1})$ of stochastic intensity process, assumed to be Markov. Let a, η, g [resp. a^*, η^*, g^*] denote the parameters of the historical dynamic [resp. risk-neutral dynamic]. Then, the transition density $\pi(y_t, p_t|y_{t-1}, p_{t-1})$ is such that:

$$\begin{aligned}
\pi(y_t, p_t | y_{t-1}, p_{t-1}) &= \pi(y_t | p_t, y_{t-1}, p_{t-1}) \pi(p_t | y_{t-1}, p_{t-1}) \text{ by Bayes theorem,} \\
&= \pi(y_t | p_t, y_{t-1}) g(p_t | p_{t-1}) \text{ by model assumptions,} \\
&= [p_t \epsilon(y_{t-1}/a p_t) + (1 - p_t) \epsilon(\frac{\eta}{1-a} + \frac{\eta}{a(1-p_t)})] g(p_t | p_{t-1}),
\end{aligned}$$

where $\epsilon(x)$ denotes a point mass at x. We get a similar decomposition for $q(y_t, p_t | y_{t-1}, p_{t-1})$. To satisfy the strict positivity of the sdf in Proposition A.1, the supports of the historical and risk-neutral distributions need to be the same. Then we deduce the following Proposition:

Proposition A.2 The historical and risk-neutral dynamics chosen in the stochastic tree family (2.1) are compatible if and only if:

$$a^* = a = exp(-r), \quad \eta^* = \eta, \quad g \text{ and } g^* \text{are strictly positive transition densities.}$$

Thus, the parameters of the tree a, η are constrained, but there is no constraint on the dynamic of the stochastic intensity.

3.3 Link with the classification of bubbles in Jarrow, Protter, Shimbo (2010)(JPS)

In a continuous time analysis JPS provide in Theorems 4.2 and 4.3 a classification of bubble assets into three categories: "Type 1 occurs when the (bubble) asset has an infinite lifetime with a payoff at $t = \infty$. Type 2 occurs when the asset's life is finite, but unbounded. Type 3 bubbles are for assets whose lives are bounded".

JPS also note on page 179 that the situation is different in discrete time where the AAO conditions differ and only Type 2 bubbles can occur, and are "uniformely integrable martingale under Q". Let us now discuss our results with respect to this classification.

3.3.1 Martingale or submartingale under Q

The stochastic tree model (2.1) is defined for $a = exp(-r) < 1$, i.e. for $r > 0$. A limiting case of martingale can be obtained when $\eta = 0, a = 1$, i.e. $r = 0$ leading to a Blanchard Watson bubble satisfying:

$$Y_t = Z_t Y_{t-1}/p_t, \quad \text{with } Z_t \sim \mathrm{B}(1, p_t)$$

This creates a (nonstationary) bubble with finite lifetime of Type 2. As in JPS, the path of the bubble asset can only feature one observed bubble pattern. In contrast, we have shown, there is a possibility of having recurrent bubble patterns if the condition of zero risk-free rate is relaxed.

3.3.2 Uniform integrability

We have pointed out that there exist stationary submartingales that are necessarily not integrable [see Proposition 3]. Therefore, by assuming the integrability, the set of bubble assets becomes artificially restricted. Generally, more equilibria of rational expectation models are obtained when the non-integrable bubbles are included among their solutions [Gourieroux, Monfort, Jasiak (2020)].