Robust Analysis of the Martingale Hypothesis
COMPLEMENTARY MATERIALS

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preliminary version
March 21, 2016

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The following Figures 1-3 present long simulated trajectories of the random walk with t-student distributed errors (degree of freedom=3), time discretized stationary diffusion and noncausal Cauchy AR(1) with autoregressive coefficient equal to 0.8.

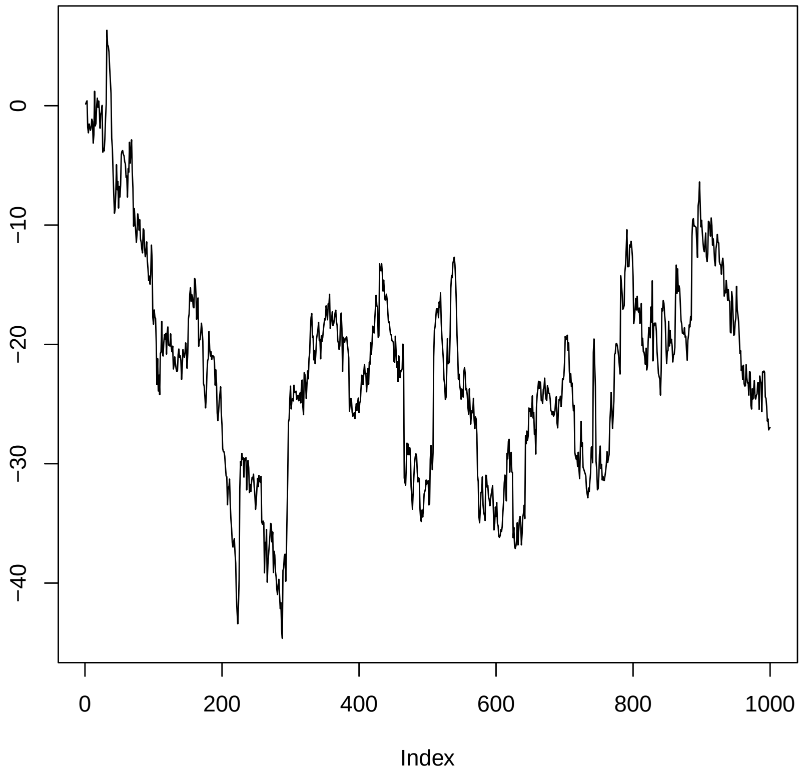

Figure 1: Random Walk

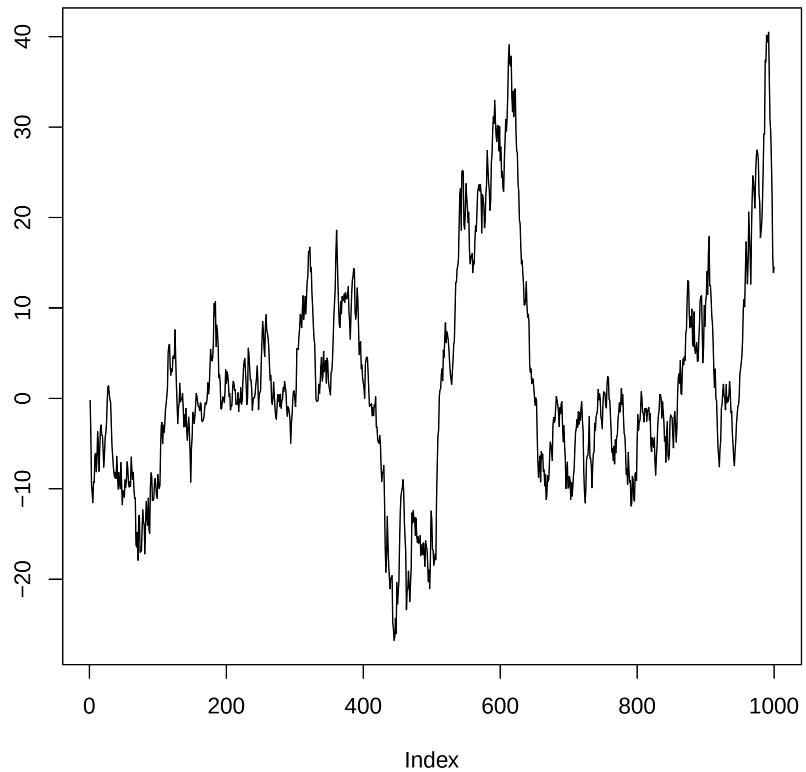

Figure 2: Time Discretized Diffusion

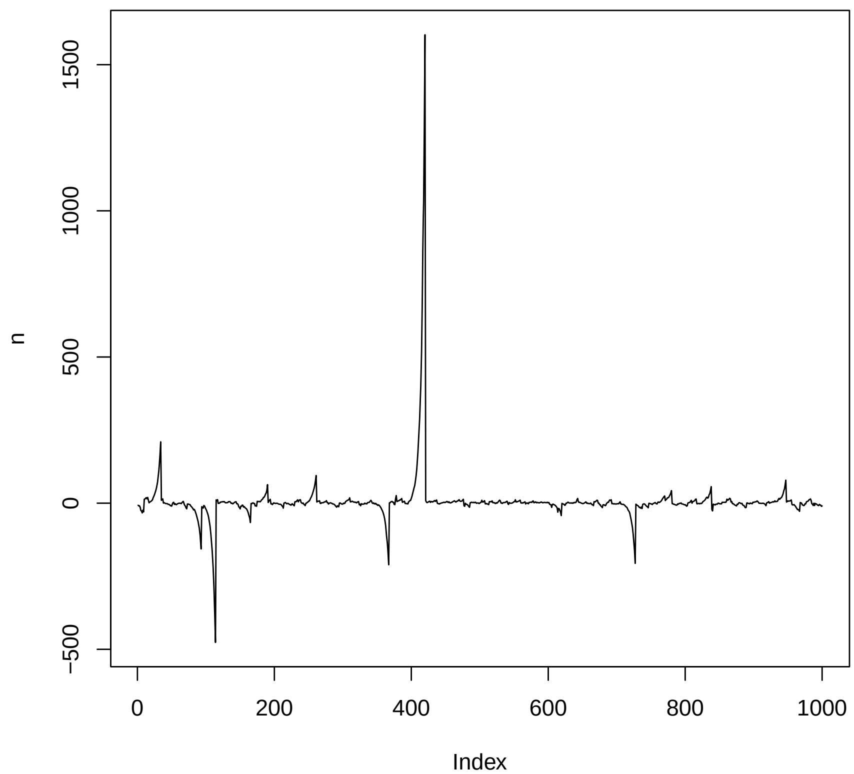

Figure 3: Noncausal Cauchy AR(1)

APPENDIX 3

Discussion of the Test Proposed in Gao, King, Lu, Tjostheim (2004)

In recent literature, there exist tests of the martingale hypothesis based on the NW estimator. Let us consider the test statistic proposed in Gao, King, Lu, Tjostheim (2009):

$$M_T = \frac{\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T} K(\frac{y_t-y_s}{h_j})\hat{u}_t\hat{u}_s}{2\sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T} K^2(\frac{y_t-y_s}{h_j})\hat{u}_t^2\hat{u}_s^2}, \tag{A.1}$$

where $\hat{u}_t = y_t - \hat{m}_t(y_{t-1})$ and the critical value is 1.96 [GKLT(2009), Th. 1.2]. In order to use the standard 1.96 critical value, an additional assumption of weak conditional heteroscedasticity (WCH) is introduced (see Appendix 1 in the main paper). Let us interpret the statistic M_T, starting from its numerator. It is a doubly localized version on a theoretical quantity of the type $\sum\sum_{a\neq t} E[(y_t - m(y_{t-1}))(y_s - m(y_{s-1}))]$. Upon an appropriate normalization, this is related to the quantity :

$$Cov[(y_t - m(y_{t-1}))(y_s - m(y_{s-1}))] = Cov(u_t, u_s) + Cov[y_{t-1} - m(y_{t-1}), y_{s-1} - m(y_{s-1})]$$

under the assumption of WCH(2009) for the null hypothesis. Thus the statistic has two components, the first one associated with the terms of the type $Cov(u_t, u_s),\ t \neq s$, is a kind of portmanteau statistic, the second one corresponding to the terms $Cov[y_{t-1} - m(y_{t-1}), y_{s-1} - m(y_{s-1})]$ is rather a measure of the difference between $m(y)$ and y.

Intuitively, this test has the same drawback as a parametric test based on the OLS estimator, despite its apparent localisation. It implicitly assumes the existence of the second-order moments, does not exactly test the form of the autoregression function and is rather a global measure. Indeed, when T is large, the sums concentrate on the pairs y_t, y_s with close values of y, giving the same weights to the extreme pairs of close values as to the other pairs of values. Thus the asymptotic properties of the test can be different from those derived in the paper in martingales with strong conditional heteroscedasticity, that is, in martingales featuring bubbles, or volatility induced mean-reversion.

We provide below the rates of acceptance of the null hypothesis in samples of various sizes T (T=200, 1000, 2000) and the following processes i) time discretized diffusion process with $\sigma(y) = \sqrt{1+|y|^{0.6}}$, ii) noncausal sutoregressive Cauchy process with $\rho^* = 0.8$, iii) Gaussian random walk, iv) Gaussian AR(1) with $\rho = 0.5$.

Table 1: Acceptance rate (in %) of the M-test

model	T=200	T=1000	T=2000
diffusion	80	90	90
noncausal	100	90	90
random walk	95	100	95
AR(1)	0	0	0