NONLINEAR CAUSALITY, with Applications to Liquidity and Stochastic Volatility

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Abstract

The conditional Laplace transform is often easier to use in financial data analysis than the conditional density. This paper characterizes nonlinear causality hypotheses for models based on the conditional Laplace transform and provides interpretations of the linear and quadratic causality in this framework. The nonlinear causality conditions are derived for a multivariate volatility model that describes volatility transmission accross national stock markets, a bivariate count model for joint trading of assets and derivatives, and a stochastic volatility model in which the drift and volatility relation is examined.

Keywords : Linear Causality, Quadratic Causality, Nonlinear Causality, Trading, Volatility Transmission, Drift-Volatility Relation, Conditional Laplace Transform, Wishart Process.

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1 Introduction

The concept of causality was introduced in econometrics for examining dynamic interactions between time series. The early methodology developed by Wiener and Granger [see, Wiener (1956), Granger (1963), (1969), Geweke (1982), Florens, Mouchart (1985), and Geweke (1984) for a survey] concerned linear dynamic macroeconomic models and investigated relationships between the conditional means of time series. For nonlinear dynamic models that became commonly used in Finance since early eighties, this methodology is inadequate as interactions between nonlinear processes involve conditional moments of higher orders.

The literature on nonlinear causality has emerged in parallel to the development of the ARCH and ARCH-related models. This is probably the reason why a predominant part of the existing literature on nonlinear causality (see e.g. Engle, Granger, Robins (1986), Cheung, Ng (1996), Comte, Liberman (2000), Hafner, Herwatz (2004)) has been focused on causality in conditional moments up to the second order. Although this new methodology is a significant extension of the early literature, it has an important limitation in that it disregards the dynamic relationships between, for example, the extremes.

An alternative approach to causality analysis that also exists in the literature relies on the conditional density and the Kullback measure of model proximity [Gourieroux, Monfort, Renault (1987)]. The conditional density-based method can account for all types of nonlinear links and accommodate all dynamic distributional properties of the time series of interest. However, it cannot be applied to nonlinear models that possess very complicated or non-tractable conditional density functions (such as Levy or multivariate compound Poisson densities, for example.). The dynamics of such series is more conveniently represented by the conditional Laplace transform. This is the case of processes in several financial applications, such as the term structure of interest rates models [Duffie, Filipovic, Schachermayer (2003), Gourieroux, Monfort, Polimenis (2006), Dai, Le, Singleton (2006), Singleton (2006)], stochastic volatility models [Gourieroux, Jasiak, Sufana (2004), Gourieroux (2006)], and more generally risk premia models [Gourieroux, Monfort (2007)].

This paper proposes a methodology for nonlinear causality analysis in multivariate conditional Laplace models and characterizes the nonlinear causality conditons. The advantage of this approach is that it accommodates all aspects of the nonlinear dynamics, even if the moments do not exist, and is feasible even when the conditional density function is not available or too complicated. For illustration, the causality conditions are derived for a multivariate volatility model that examines the volatility transmission, a bivariate count model for stock and derivative trading and a stochastic volatility model in which the drift- volatility relationship is studied.

The paper is organized as follows. Section 2 presents the characterization of nonlinear noncausality from the conditional log-Laplace transform, and provides the definitions of linear and quadratic noncausality based on the Taylor series expansion of the conditional Laplace transform. This approach is applied to the class of Compound Autoregressive (CaR) processes and to their recursive specifications in Section 3. Section 4 concerns the transmission of volatilities between national stock markets, and the possibility of disregarding the covolatility effects in the analysis. Section 5 examines nonlinear causality in a dynamic bivariate count model for joint trading of an asset and its derivatives. Section 6 investigatess the relationship between the drift and stochastic volatility. It emphasizes the role of the information set in causality analysis. Section 7 concludes. Proofs are gathered in Appendices.

2 Characterizations of noncausality

Let us consider a multivariate process $x_t = (x'_{1,t}, x'_{2,t})'$. The component series $(x_{1,t})$ and $(x_{2,t})$ are of dimensions n_1 and n_2, respectively. (x_t) is assumed to be a Markov process or order one [3] with the conditional log-Laplace transform:

$$\psi_t(u) = \psi_t(u_1, u_2) = \log E[\exp(u'_1 x_{1,t+1} + u'_2 x_{2,t+1}) | x_{1,t}, x_{2,t}],$$

for $u \in D$, where D is the domain of existence of the expectation of exponential transforms of X_t.

2.1 Nonlinear causality

The nonlinear noncausality can be unidirectional or instantaneous, according to the traditional terminology introduced by Granger(1969). These two forms of nonlinear causality are considered below and defined from the conditional log-Laplace transform of the series of interest. The following definitions are

[3] The extension to Markov process of any autoregressive order p is straightforward.

equivalent to the conditional density definitions of nonlinear causality given in Gourieroux, Monfort, Renault (1987).

Definition 1 :
i) **Instantaneous noncausality**
x_1 does not (nonlinearly) instantaneously cause x_2, iff $x_{1,t+1}$ and $x_{2,t+1}$ are independent conditional on $x_{1,t}, x_{2t}$, or equivalently if the log-Laplace transform can be written as:

$$\psi_t(u_1, u_2) = \psi_{1,t}(u_1) + \psi_{2,t}(u_2),$$

where $\psi_{j,t}(u_j)$ is a function of $u_j, x_{1,t}, x_{2,t}$.

ii) **Unidirectional noncausality from x_1 to x_2**

x_1 does not (nonlinearly) cause x_2 iff the conditional distribution of $x_{2,t+1}$ given $x_{1,t}, x_{2,t}$ does not depend on $x_{1,t}$. Equivalently, the log-Laplace transform $\psi_t(0, u_2)$ does not depend on $x_{1,t}$.

iii) **Unidirectional noncausality from x_2 to x_1**

x_2 does not (nonlinearly) cause x_1, iff $\psi_t(u_1, 0)$ does not depend on $x_{2,t}$.

2.2 First- and Second- Order Causality

In the literature, we commonly find references to weaker notions of causality limited to the first- and second- order conditional moments. This section provides a new log-Laplace transform-based interpretation of the first- and second- order causality.

(a) First-order (linear) Causality

The first-order conditional moment is equal to the first-order derivative of the conditional log-Laplace transform evaluated at zero :

$$E\left[x_{t+1}|x_t\right] = \frac{\partial \psi_t}{\partial u}(0).$$

Hence, to find the log-Laplace equivalent of the linear causality conditions, we consider the first-order expansion of the log-Laplace transform evaluated at zero[4] :

$$\psi_t^{(1)}(u_1, u_2) = u_1' \frac{\partial \psi_t}{\partial u_1}(0,0) + u_2' \frac{\partial \psi_t}{\partial u_2}(0,0).$$

$\psi_t^{(1)}$ is the log-Laplace transform of a point mass distribution with a single admissible value equal to the conditional expectation $E(x_{t+1}|x_t) = \dfrac{\partial \psi_t}{\partial u}(0)$.

Definition 2 :
The first-order (linear) noncausality is the nonlinear noncausality implied by the first-order expansion of ψ_t.

The Proposition below summarizes the conditions of instantaneous and unidirectional noncausalities.

Proposition 1 :
i) (x_1) does not instantaneously cause (x_2) at first-order.
ii) (x_1) does not cause (x_2) at first-order, iff $E(x_{2,t+1}|x_{1,t}, x_{2,t})$ does not depend of $x_{1,t}$, or equivalently if : $\dfrac{\partial \psi_t}{\partial u_2}(0,0)$ does not depend on $x_{1,t}$.
iii) (x_2) does not cause (x_1) at first-order, iff $E(x_{1,t+1}|x_{1,t}, x_{2,t})$ does not depend on $x_{2,t}$, or equivalently : $\dfrac{\partial \psi_t}{\partial u_1}(0,0)$ does not depend on $x_{2,t}$.

As in the initial definition by Granger (1969), (1988), the instantaneous causality at first-order does not exist.

(b) Second-Order (quadratic) Causality

To find the log-Laplace equivalents of the conditions of second-order causality, we need to examine the second-order expansion of the log-Laplace transform. Since $V(x_{t+1}|x_t) = \dfrac{\partial^2 \psi_t}{\partial u \partial u'}(0)$, the second-order expansion of the log-Laplace transform is :

[4]Note that $\psi_t(0,0) = 0$.

$$
\begin{aligned}
\psi_t^{(2)}(u_1, u_2) &= u_1' \frac{\partial \psi_t}{\partial u_1}(0,0) + u_2' \frac{\partial \psi_t}{\partial u_2}(0,0) + \frac{1}{2} u' \frac{\partial^2 \psi_t(0,0)}{\partial u \partial u'} u \\
&= u_1' E(x_{1,t+1}|x_{1,t}, x_{2,t}) + u_2' E(x_{2,t+1}|x_{1,t}, x_{2,t}) + \frac{1}{2} u' V(x_{t+1}|x_t) u.
\end{aligned}
$$

$\psi_t^{(2)}$ is the log-Laplace transform of a Gaussian distribution with mean $E(x_{t+1}|x_t)$ and variance-covariance matrix $V(x_{t+1}|x_t)$. This Gaussian distribution provides the best second-order approximation to the conditional distribution of interest (in terms of the Laplace transform).

Definition 3 :
The second-order noncausality is the nonlinear noncausality implied by the second-order expansion of ψ_t.

The conditions of instantaneous and unidirectional second-order noncausalities are given in the following proposition.

Proposition 2 :
i) (x_1) does not instantaneously cause (x_2) at second-order, iff :
$\frac{\partial^2 \psi_t}{\partial u_1 \partial u_2}(0,0) = 0 \Longleftrightarrow Cov(x_{1,t+1}, x_{2,t+1}|x_{1,t}, x_{2,t}) = 0.$
ii) (x_1) does not cause (x_2) at second-order, iff : $\frac{\partial \psi_t}{\partial u_2}(0,0)$ and $\frac{\partial^2 \psi_t}{\partial u_2 \partial u_2'}(0,0)$ do not depend on $x_{1,t}$, or equivalently, iff : $E(x_{2,t+1}|x_{1,t}, x_{2,t})$ and $V(x_{2,t+1}|x_{1,t}, x_{2,t})$ do not depend on $x_{1,t}$.
iii) (x_2) does not cause (x_1) at second-order, iff $\frac{\partial \psi_t}{\partial u_1}(0,0)$ and $\frac{\partial^2 \psi_t}{\partial u_1 \partial u_1'}(0,0)$ do not depend on $x_{2,t}$, or equivalently, iff : $E(x_{1,t+1}|x_{1,t}, x_{2,t})$ and $V(x_{1,t+1}|x_{1,t}, x_{2,t})$ do not depend on $x_{2,t}$.

This characterization is related to the definition of noncausality in variance introduced in Comte, Liberman (2000) [5], who show that unidirectional noncausality in variance (noncausality-in-variance) implies noncausality in mean. Therefore, x_1 does not cause x_2 at second-order, iff $V(x_{2,t+1}|x_{1,t}, x_{2,t})$ does not depend on $x_{1,t}$.

[5] See the discussion in Caperin (2003) for a comparison of second-order noncausality to noncausality-in-variance introduced in Engle, Granger, Robins (1986).

The nonlinear noncausality of a given type implies the corresponding second-order noncausality, which in turn implies the first-order noncausality of the same type. This property can be used for a sequential causality analysis: In the first step, the hypothesis of first-order noncausality is tested. If this hypothesis is not rejected, then the hypothesis of second-order noncausality is considered. If it is not rejected, the test of nonlinear noncausality should follow, etc.

3 Causality in CaR Processes

The models of multivariate volatility, bivariate count, and stochastic volatility used in causality analysis later in the text, as well as several models used recently in the term structure analysis, belong to the family of compound autoregressive (CaR) processes [see Darolles, Gourieroux, Jasiak (2006)]. The CaR processes are defined from the Laplace transforms, and arise as nonlinear dynamic extensions of Gaussian VAR models. This section provides the noncausality conditions for compound autoregressive (CaR) processes and their recursive specifications that in some cases can be more convenient for econometric analysis.

3.1 Compound autoregressive processes

The CaR processes [6] [see Darolles, Gourieroux, Jasiak (2006)] have conditional log-Laplace transforms that are affine functions of the conditioning variables :

$$\begin{aligned} \psi_t(u) &= a'(u)x_t + b(u) \\ &= a_1'(u_1,u_2)x_{1,t} + a_2'(u_1,u_2)x_{2,t} + b(u_1,u_2). \end{aligned}$$

The noncausality conditions concern functions a and b.

Proposition 3:
For a CaR process with twice differentiable functions a_1, a_2, b, we have the following noncausality conditions:

[6] Also called affine processes in the continuous time financial literature [Duffie, Filipovic, Schachermayer (2003)].

i) **Instantaneous noncausality**

$$\text{nonlinear}: \quad a(u_1, u_2) = a_1(u_1) + a_2(u_2), \ (\text{with } a_1(0) = a_2(0) = 0), \text{ and}$$

$$b(u_1, u_2) = b_1(u_1) + b_2(u_2), \ (\text{with } b_1(0) = b_2(0) = 0);$$

at first-order: always;
at second-order: $\frac{\partial^2 a}{\partial u_1 \partial u_2'}(0,0) = 0$, and $\frac{\partial^2 b}{\partial u_1 \partial u_2'}(0,0) = 0$.

ii) **Unidirectional noncausality from x_1 to x_2**

nonlinear : $a_1(0, u_2) = 0$;
at first-order: $\frac{\partial a_1}{\partial u_2}(0,0) = 0$;
at second-order: $\frac{\partial a_1}{\partial u_2}(0,0) = 0$, and $\frac{\partial^2 a_1}{\partial u_2 \partial u_2'}(0,0) = 0$.

iii) **Unidirectional noncausality from x_2 to x_1**

nonlinear : $a_2(u_1, 0) = 0$;
at first-order: $\frac{\partial a_2}{\partial u_1}(0,0) = 0$;
at second-order: $\frac{\partial a_2}{\partial u_1}(0,0) = 0$, and $\frac{\partial^2 a_2}{\partial u_1 \partial u_1'}(0,0) = 0$.

3.2 Recursive CaR Specification

The CaR models can be specified in a recursive form. A typical example is the Heston's model of stochastic volatility [see Heston (1993) and Section 6]. This section examines a recursive CaR specification defined in the two following steps. First, the conditional Laplace transform of $x_{1,t+1}$ given $x_{2,t+1}, x_{1,t}, x_{2,t}$ is defined as:

$$E[\exp(u_1' x_{1,t+1}) | x_{2,t+1}, x_{1,t}, x_{2,t}] = \exp[\alpha_t(u_1)' x_{2,t+1} + \beta_t(u_1)], \tag{2.1}$$

where α_t, β_t are path dependent functions. This expression assumes that the conditional log-Laplace transform is affine with respect to $x_{2,t+1}$.

Next, the conditional Laplace transform of $x_{2,t+1}$ given $x_{1,t}, x_{2,t}$ is considered:

$$E\left[\exp(u_2' x_{2,t+1}) | x_{1,t}, x_{2,t}\right] = \exp \gamma_t(u_2), \tag{2.2}$$

where γ_t is a path-dependent function as well.

The joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ follows from the iterated expectation theorem:

$$\begin{aligned}
& \exp \psi_t(u_1, u_2) \\
= \; & E\left[\exp\left(u_1' x_{1,t+1} + u_2' x_{2,t+1}\right) | x_{1,t}, x_{2,t}\right] \\
= \; & E\left\{E\left[\exp(u_1' x_{1,t+1}) | x_{2,t+1}, x_{1,t}, x_{2,t}\right] \exp(u_2' x_{2,t+1}) | x_{1,t}, x_{2,t}\right\} \\
= \; & \exp\left\{\gamma_t[\alpha_t(u_1) + u_2] + \beta_t(u_1)\right\}. \qquad (2.3)
\end{aligned}$$

The noncausality conditions for the recursive CaR specification are based on Definition 1 and Propositions 1-3 [see Appendix 1].

Proposition 4:
For the recursive CaR specification with twice differentiable functions $\alpha_t, \beta_t, \gamma_t$, the noncausality conditions are:

i) **Instantaneous noncausality**

nonlinear : $\alpha_t = 0$, or γ_t linear (that is, x_2 conditionally deterministic);
at first-order: always;
at second-order: $\dfrac{\partial^2 \gamma_t}{\partial u_2 \partial u_2'}(0) \dfrac{\partial \alpha_t}{\partial u_1'}, (0) = 0$.

ii) **Unidirectional noncausality from x_1 to x_2**

nonlinear : $\gamma_t(u_2)$ does not depend on $x_{1,t}$;
at first-order: $\dfrac{\partial \gamma_t}{\partial u_2}(0)$ does not depend on $x_{1,t}$;
at second-order: $\dfrac{\partial \gamma_t}{\partial u_2}(0)$ and $\dfrac{\partial^2 \gamma_t(0)}{\partial u_2 \partial u_2'}$ do not depend on $x_{1,t}$

iii) **Unidirectional noncausality from x_2 to x_1**

nonlinear : $\gamma_t[\alpha_t(u_1)] + \beta_t(u_1)$ does not depend on $x_{2,t}$;

at first-order: $\frac{\partial \gamma_t}{\partial u_2'}(0)\frac{\partial \alpha_t}{\partial u_1'}(0) + \frac{\partial \beta_t}{\partial u_1'}(0)$ does not depend on $x_{2,t}$;

at second-order: $\frac{\partial \gamma_t}{\partial u_2'}(0)\frac{\partial \alpha_t}{\partial u_1'}(0) + \frac{\partial \beta_t}{\partial u_1'}(0)$, and

$\sum_{j=1}^{n_2}[\frac{\partial \gamma_t}{\partial u_{2j}}(0)\frac{\partial^2 \alpha_{jt}}{\partial u_1 \partial u_1'}(0)] \quad + \quad \sum_{j=1}^{n_2}[\frac{\partial \alpha_{jt}}{\partial u_1}(0)\frac{\partial^2 \gamma_t}{\partial u_{2j}\partial u_2'}(0)]\frac{\partial \alpha_t}{\partial u_1'}(0) \quad + \quad \frac{\partial^2 \beta_t}{\partial u_1 \partial u_1'}(0)$

do not depend on $x_{2,t}$

4 Volatility Transmission Accross Markets

There exists a large body of literature [see, e.g. Von Furstenberg, Nam Jeon (1989), Roll (1989), King, Wadhwani (1990), Hamao, Masulis, Ng (1990), King, Sentana, Wadhwani (1994)] on volatility transmission accross national markets. A common approach consists in considering the returns on two market indexes, such as the S&P and FTSE, for example, expressed in the same currency, such as the US Dollar. The volatilities of both markets are then computed, and noncausality tests are performed. However, this approach is valid only if the conditional correlation between the returns on market indexes has no effect on their volatilities. Thus, before considering the causality between volatilities, it is necessary to test for noncausality between the series of correlations and the two market volatility processes. This approach is illustrated below in the framework of a Wishart autoregressive (WAR) process for volatility-covolatility matrices [Gourieroux (2006), Gourieroux, Jasiak, Sufana (2007)], which belongs to the family of CaR processes.

4.1 The WAR process for stochastic volatility-covolatility

Let us denote the volatility and covolatility process by

$$Y_t = \begin{bmatrix} Y_{11,t} & Y_{12,t} \\ Y_{12,t} & Y_{22,t} \end{bmatrix},$$

where $Y_{11,t}$ and $Y_{22,t}$ are the volatilities of markets 1 and 2, respectively, and $Y_{12,t}$ is the covolatility. The volatility-covolatility process Y_t is a WAR process of order one with the conditional Laplace transform of Y_{t+1}:

$$
\begin{aligned}
\Psi_t(\Gamma) &= E[\exp Tr(\Gamma Y_{t+1})|Y_t] \\
&= E\left[\exp\left(\gamma_{11}Y_{11,t+1}+\gamma_{22}Y_{22,t+1}+2\gamma_{12}Y_{12,t+1}\right)|Y_t\right] \\
&= \frac{\exp Tr\left[M'\Gamma(Id-2\Sigma\Gamma)^{-1}MY_t\right]}{\left[\det(Id-2\Sigma\Gamma)\right]^{K/2}},
\end{aligned}
$$

where Γ is a 2×2 symmetric matrix of arguments of the Laplace transform, Tr is the trace operator, and γ_{ij} and $Y_{ij,t+1}$ the ij^{th} element of Γ and Y_{t+1} respectively. M, Σ and K are the parameters: M is a 2×2 matrix, Σ is a 2×2 symmetric matrix of full rank, and K is the scalar degree of freedom assumed greater or equal to 2: $K \geq 2$.

In the remainder of this section, we are interested in nonlinear causality from stochastic correlation $Y_{12,t}(Y_{11,t}Y_{22,t})^{-1/2}$ to volatilities $(Y_{11,t},Y_{22,t})$ [see, Jasiak, Lu (2006) for a general discussion of noncausality in WAR]. In a nonlinear framework, this noncausality is equivalent [7] to the nonlinear noncausality from covolatility $Y_{12,t}$ to volatilities $(Y_{11,t},Y_{22,t})$.

4.2 The causality condition

The following proposition is proven in Appendix 2 by considering the conditional Laplace transform of $(Y_{11,t},Y_{22,t})$ given by

$$
\begin{aligned}
&\Psi_t(\gamma_{11},\gamma_{22}) \\
&= \frac{\exp Tr\left[M'\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}\left[Id-2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}\right]^{-1} MY_t\right]}{\det\left[Id-2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}\right]^{K/2}}.
\end{aligned}
$$

Proposition 5: In the WAR framework, covolatility $Y_{12,t}$ does not cause nonlinearly volatilities $(Y_{11,t},Y_{22,t})$, if and only if, one of the following conditions is satisfied:

i) $m_{11}=m_{21}=0$;

ii) $m_{12}=m_{22}=0$;

[7] The possibility to replace the correlation by the covolatility is only relevant for nonlinear noncausality, and not for first-, or second-order noncausality.

iii) $m_{12} = m_{21} = \sigma_{12} = 0$;
iv) $m_{11} = m_{22} = \sigma_{12} = 0$,
where m_{ij} (resp. σ_{ij}) denote the elements of matrix M (resp. Σ).

Thus, using clear notation, the hypothesis of noncausality can be written as the union $H_{i)} \cup H_{ii)} \cup H_{iii)} \cup H_{iv)}$, where the four elementary hypotheses are defined by the equality constraints on the parameters. Let us now discuss the four elementary hypotheses.

Hypothesis $H_{i)}$: $m_{11} = m_{21} = 0$,

Under $H_{i)}$, the quantity

$$Tr\left[M'\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}[Id - 2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}]^{-1}MY_t\right]$$

is proportional to $Y_{22,t}$. Thus, the current value of volatility $Y_{22,t}$ is driving both future volatilities.

Hypothesis $H_{ii)}$: $m_{12} = m_{22} = 0$.

This is a symmetric case with $Y_{11,t}$ as the driving variable.

Hypothesis $H_{iii)}$: $m_{12} = m_{21} = \sigma_{12} = 0$.

Under $H_{iii)}$, the trace in the expression of the conditional Laplace Transform $\Psi_t(\gamma_{11}, \gamma_{22})$ becomes

$$\frac{m_{11}^2\gamma_{11}Y_{11t}}{1 - 2\sigma_{11}\gamma_{11}} + \frac{m_{22}^2\gamma_{22}Y_{22t}}{1 - 2\sigma_{22}\gamma_{22}},$$

which is a sum of functions of $(\gamma_{11}, Y_{11,t})$ and $(\gamma_{22}, Y_{22,t})$. There is noncausality between the volatilities in the sense that $(Y_{11,t})$ does not cause $(Y_{22,t})$ and $(Y_{22,t})$ does not cause $(Y_{11,t})$.

Hypothesis $H_{iv)}$: $m_{11} = m_{22} = \sigma_{12} = 0$,

The trace term in the expression of the conditional Laplace Transform $\Phi_t(\gamma_1, \gamma_2)$ becomes

$$\frac{m_{21}^2\gamma_{22}Y_{11t}}{1 - 2\sigma_{22}\gamma_{22}} + \frac{m_{12}^2\gamma_{11}Y_{22t}}{1 - 2\sigma_{11}\gamma_{11}}.$$

Thus, under hypothesis $H_{iv)}$, the volatilities $Y_{11,t+1}$ and $Y_{22,t+1}$ are conditionally independent. Moreover, the volatility $Y_{11,t+1}$ (resp. $Y_{22,t+1}$) depends on the past through $Y_{22,t}$ (resp. $Y_{11,t}$) only. There is a periodic pattern of period 2 in the causality between volatilities, since at horizon 2, $Y_{11,t+2}$ (resp.

$Y_{22,t+2}$) depends on $Y_{11,t}$ (resp. $Y_{22,t}$). This case illustrates volatility feedback between stock markets.

The above discussion is summarized in Table 1.

Table 1 : Noncausality from covolatility to volatilities

hypothesis	$H_{i)}$	$H_{ii)}$	$H_{iii)}$	$H_{iv)}$
conditions	$m_{11}=0$ $m_{21}=0$	$m_{12}=0$ $m_{22}=0$	$m_{12}=0$ $m_{21}=0$ $\sigma_{12}=0$	$m_{11}=0$ $m_{22}=0$ $\sigma_{12}=0$
interpretation	$(Y_{22,t})$ is driving	$(Y_{11,t})$ is driving	$(Y_{11,t})$ doesn't cause $(Y_{22,t})$ and $(Y_{22,t})$ doesn't cause $(Y_{11,t})$	volatility feedback

5 Trading Analysis

5.1 The model

The analysis of trading processes on financial markets is commonly based on data such as the daily counts of trades, or the intertrade durations. In this section, we consider two types of assets, i.e. a stock and one derivative written on this stock, and focus on the daily counts of trades. Thus, $X_{1,t}$ [resp. $X_{2,t}$] denotes the daily count of trades of the underlying asset [resp. the derivative].

A dynamic Compound Poisson model is constructed by considering the trade intensities of the two assets, and the count of market events, which can (but not necessarily does) generate trades. More precisely, we assume:

$$X_{1,t+1} = \sum_{i=1}^{Z_{t+1}} Y_{1,i,t+1}, \quad X_{2,t+1} = \sum_{i=1}^{Z_{t+1}} Y_{2,i,t+1}, \tag{5.1}$$

where the conditional distribution of the count of market events Z_{t+1} is Poisson $\mathcal{P}[\mu + \lambda_1 X_{1,t} + \lambda_2 X_{2,t}]$. The marks $(Y_{1,i,t+1}, Y_{2,i,t+1})$ are assumed independent, identically distributed and independent of Z_{t+1} conditional on $X_{1,t}, X_{2,t}$. The variables Y can take the values 0 and 1 and have the following joint distribution: $P[Y_{1,i,t} = k, Y_{2,i,t} = l] = p_{kl},\ k = 0, 1,\ l = 0, 1.$

At each new market event i, we can observe either trades of the two types of assets when $k = l = 1$, or trades of a single type of asset, such as the underlying asset $(k = 1, l = 0)$ or a derivative $(k = 0, l = 1)$, or no trades at all $(k = 0, l = 0)$.

Let us now derive the log-Laplace transform of the two count variables X_1, X_2. We get:

$$
\begin{aligned}
&E_t[exp(u_1 X_{1,t+1} + u_2 X_{2,t+1})] \\
&= E_t \left(exp \left[\sum_{i=1}^{Z_{t+1}} (u_1 Y_{1,i,t+1} + u_2 Y_{2,i,t+1}) \right] \right) \\
&= E_t E \left[exp \sum_{i=1}^{Z_{t+1}} (u_1 Y_{1,i,t+1} + u_2 Y_{2,i,t+1}) | X_{1,t}, X_{2,t}, Z_{t+1} \right] \\
&= E_t \left([E exp(u_1 Y_{1,i} + u_2 Y_{2,i})]^{Z_{t+1}} \right) \\
&= E_t \left[(p_{11} exp(u_1 + u_2) + p_{1,0} exp(u_1) + p_{0,1} exp(u_2) + p_{0,0})^{Z_{t+1}} \right] \\
&= E_t \left[(1 - p_{1,1}(1 - exp(u_1 + u_2)) - p_{1,0}(1 - \exp u_1) - p_{0,1}(1 - \exp u_2)]^{Z_{t+1}} \right].
\end{aligned}
$$

By using the expression of the Laplace transform for the Poisson variable Z_{t+1}, we find:

$$
\begin{aligned}
&E_t[exp(u_1 X_{1,t+1} + u_2 X_{2,t+1})] \\
&= exp[-(\mu + \lambda_1 X_{1,t} + \lambda_2 X_{2,t})\{p_{1,1}[1 - exp(u_1 + u_2)] + p_{1,0}(1 - expu_1) + p_{0,1}(1 - expu_2)\}]
\end{aligned}
$$

and

$$
\begin{aligned}
\Psi_t(u_1, u_2) &= -(\mu + \lambda_1 X_{1,t} + \lambda_2 X_{2,t})[p_{1,1}(1 - exp(u_1 + u_2)] \\
&+ p_{1,0}(1 - expu_1) + p_{0,1}(1 - expu_2)]]. \qquad (5.2)
\end{aligned}
$$

Since the conditional log-Laplace transform is affine in $X_{1,t}, X_{2,t}$, the model is a CaR process. The first- and second-order conditional moments of the count variables are:

$$
\begin{aligned}
E_t(X_{t+1}) &= E_t[X_{t+1}|Z_{t+1}, X_t] \\
&= E_t\left[Z_{t+1}\begin{pmatrix} p_{1.} \\ p_{.1} \end{pmatrix}\right], \\
&= (\mu + \lambda_1 X_{1,t} + \lambda_2 X_{2,t})\begin{pmatrix} p_{1.} \\ p_{.1}, \end{pmatrix} \qquad (5.3)
\end{aligned}
$$

where $X_t = (X_{1,t}, X_{2,t})'$, $p_{1.} = p_{1,1} + p_{1,0}$, $p_{.1} = p_{1,1} + p_{0,1}$.

$$
\begin{aligned}
V_t(X_{t+1}) &= E_t V[X_{t+1}|Z_{t+1}, X_t] + V_t E[X_{t+1}|Z_{t+1}, X_t] \\
&\quad \text{(by the variance decomposition formula)} \\
&= E_t\left[Z_{t+1}\begin{pmatrix} p_{1.} - p_{1.}^2 & p_{1,1} - p_{1.}p_{.1} \\ p_{1,1} - p_{1.}p_{.1} & p_{.1} - p_{.1}^2 \end{pmatrix}\right] + V_t\left[Z_{t+1}\begin{pmatrix} p_{1.} \\ p_{.1} \end{pmatrix}\right] \\
&= (\mu + \lambda_1 X_{1,t} + \lambda_2 X_{2,t})\begin{pmatrix} p_{1.} & p_{1,1} \\ p_{1,1} & p_{.1} \end{pmatrix}. \qquad (5.4)
\end{aligned}
$$

5.2 Noncausality analysis

The noncausality conditions for the dynamic Compound Poisson model are based on the closed-form expression of the conditional log-Laplace transform derived in the previous section.

Proposition 6: For the dynamic Compound Poisson model, the noncausality hypotheses are characterized as follows:

i) **Instantaneous noncausality**
nonlinear=second-order : $p_{11} = 0$; first-order: always.

ii) **Unidirectional noncausality from x_1 to x_2:**
nonlinear=second-order=first-order: $(\lambda_1 = 0)$, or $(p_{1,1} = p_{0,1} = 0)$.

iii) **Unidirectional noncausality from x_2 to x_1:**
nonlinear=second-order=first-order: $(\lambda_2 = 0)$, or $(p_{1,1} = p_{1,0} = 0)$.

As in the Gaussian VAR model, the second-order and nonlinear causality conditions are equivalent. There is instantaneous noncausality, iff simultaneous trades of the two types of assets are not allowed. The test of instantaneous noncausality hypothesis is of special importance. Indeed, the assumption of simultaneous trading underlies the standard no-arbitrage argument, used to compute derivative prices without ambiguity. The unidirectional causality hypotheses are important as their tests can reveal whether the underlying asset is driving its derivative markets, or vice-versa. Finally, note that the unidirectional noncausality hypothesis from x_1 to x_2 can be written as the union $(\lambda_1 = 0) \cup (p_{1,1} = p_{0,1} = 0)$ of two elementary hypotheses defined by equality constraints on parameters.

6 The drift-volatility relationship

According to the financial literature, there exists a positive relationship between expected returns and volatility [see,e.g. Glosten, Jaganathan, Runkle (1993)], viewed as a consequence of the Capital Asset Pricing Model (CAPM) [Merton (1974)]. For empirical researchers, however, the existence and sign of this relationship is unclear. In particular, when the effects of lagged variables are taken into account, that is, when the drift includes both the current and lagged volatilities, the sensitivity coefficient on current volatility is often negative, while the sum of sensitivity coefficients on both volatilities is positive. In some sense, we get a negative relationship in the short run and a positive relationship in the long run, called the volatility feedback (Bekaert, Wu (2000)). In addition, the notions of drift and volatility depend on the conditioning set (i.e. the information set), and so does the relationship between the drift and volatility. For example, when the drift and volatility are defined by an ARCH-in-mean model, the drift series (resp. the volatility series) contains the same information as the return series (resp. the series of square returns). Therefore, there exist strong instantaneous nonlinear relationships between them. On the contrary, in the absence of information, the (unconditional) drift and volatility are constant and independent of one another.

The aim of this section is to discuss the first-, second-order and nonlinear causality between the drift and volatility (resp. return and volatility) in the framework of a stochastic volatility model. This framework has been chosen to avoid the deterministic nonlinear instantaneous relationship that exists in ARCH models.

6.1 The model

Let us consider a stochastic volatility model for the short term interest rate in discrete time. The model is defined recursively: we first specify the conditional distribution of interest rate r_{t+1} given the lagged values of interest rate $\underline{r_t}$ and the current and lagged values of the volatility factor $\underline{\sigma^2_{t+1}}$. Then, we specify the conditional distribution of σ^2_{t+1} given $\underline{r_t}, \underline{\sigma^2_t}$. These conditional distributions are characterized by their Laplace transforms and are autoregressive gamma (ARG) to ensure the positivity of both r_{t+1} and σ^2_{t+1} [see Gourieroux, Jasiak (2005) for the definition and description of autoregressive gamma processes]. We have the following recursive CaR specification:

$$E\left[\exp(-u_1 r_{t+1})|\underline{r_t}, \underline{\sigma^2_{t+1}}\right] = \exp\left[-\delta \log(1+cu_1) - \frac{cu_1}{1+cu_1}(\beta_0 r_t + \beta_1 \sigma^2_t + \gamma \sigma^2_{t+1})\right],$$

$$E\left[\exp(-u_2 \sigma^2_{t+1})|\underline{r_t}, \underline{\sigma^2_t}\right] = \exp\left[-\delta^* \log(1+c^* u_2) - \frac{c^* u_2}{1+c^* u_2}(\delta_0 r_t + \delta_1 \sigma^2_t)\right],$$

where the parameters are constrained by :

$$\begin{aligned} \delta > 0, c > 0, \delta^* > 0, c^* > 0, \\ \beta_0 \geq 0, \beta_1 \geq 0, \gamma \geq 0, \delta_0 \geq 0, \delta_1 \geq 0. \end{aligned} \quad (5.5)$$

By considering the second-order expansion of the log-Laplace transforms, we get the associated first-and second- order conditional moments :

$$\left\{ \begin{array}{l} E\left(r_{t+1}|\ \underline{r_t}, \underline{\sigma^2_{t+1}}\right) = c(\delta + \beta_0 r_t + \beta_1 \sigma^2_t + \gamma \sigma^2_{t+1}), \\ V\left(r_{t+1}|\ \underline{r_t}, \underline{\sigma^2_{t+1}}\right) = c^2[\delta + 2(\beta_0 r_t + \beta_1 \sigma^2_t + \gamma \sigma^2_{t+1})], \end{array} \right.$$

$$\left\{ \begin{array}{l} E\left(\sigma^2_{t+1}|\ \underline{r_t}, \underline{\sigma^2_t}\right) = c^* \left(\delta^* + \delta_0 r_t + \delta_1 \sigma^2_t\right), \\ V\left(\sigma^2_{t+1}|\ \underline{r_t}, \underline{\sigma^2_t}\right) = c^{*2}\left[\delta^* + 2(\delta_0 r_t + \delta_1 \sigma^2_t)\right]. \end{array} \right.$$

Conditional on the information set : $I_t = \left(\underline{r_t}, \underline{\sigma^2_{t+1}}\right)$, the drift and volatility of the interest rate satisfy a deterministic positive affine relationship. This deterministic relationship is eliminated by considering a slightly smaller information set : $J_t = \left(\underline{r_t}, \underline{\sigma^2_t}\right)$.

6.2 Joint conditional distribution of $(r_{t+1}, \sigma^2_{t+1})$

The conditional distribution of $(r_{t+1}, \sigma^2_{t+1})$ can be defined by the following joint Laplace transform [see, Appendix 3, i)]:

$$E\left[\exp(-u_1 r_{t+1} - u_2 \sigma^2_{t+1}) | \underline{r_t}, \underline{\sigma^2_t}\right] = \exp \psi_t(u_1, u_2), \text{ say},$$

where

$$\begin{aligned}
\psi_t(u_1, u_2) &= -\delta \log(1 + cu_1) - \delta^* \log\left[1 + c^*(u_2 + \frac{\gamma c u_1}{1 + cu_1})\right] \\
&- r_t \left\{ \beta_0 \frac{cu_1}{1 + cu_1} + \frac{\delta_0 c^* \left[u_2 + \frac{\gamma c u_1}{1 + cu_1}\right]}{1 + c^* \left[u_2 + \frac{\gamma c u_1}{1 + cu_1}\right]} \right\} \\
&- \sigma^2_t \left\{ \beta_1 \frac{cu_1}{1 + cu_1} + \frac{\delta_1 c^* \left[u_2 + \frac{\gamma c u_1}{1 + cu_1}\right]}{1 + c^* \left[u_2 + \frac{\gamma c u_1}{1 + cu_1}\right]} \right\}.
\end{aligned}$$

The conditional log-Laplace transform is an affine function of (r_t, σ^2_t). Therefore, the joint process (r_t, σ^2_t) is a CaR process and the results of Section 3 can be applied. In particular, the drift and volatility of the short term interest rate, conditional on the smaller information set $J_t = (\underline{r_t}, \underline{\sigma^2_t})$ are affine functions of the lagged values. By considering the second-order expansion of $\psi_t(u_1, 0)$, that is, $\psi_t(u_1, 0) = -u_1 \mu_t + \frac{u_1^2}{2} \eta^2_t + o(u_1^2)$, we get [see Appendix 3 ii)] :

$$
\begin{aligned}
\mu_t &= E\left(r_{t+1}|\underline{r_t}, \underline{\sigma_t^2}\right) \\
&= c(\delta + c^*\gamma\delta^*) + cr_t(\beta_0 + c^*\gamma\delta_0) + c\sigma_t^2(\beta_1 + c^*\gamma\delta_1), \\
\eta_t^2 &= V(r_{t+1}|\underline{r_t}, \underline{\sigma_t^2}) \\
&= c^2(\delta + 2c^*\gamma\delta^* + c^{*2}\gamma^2\delta^*) \\
&+ 2c^2 r_t(\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0) \\
&+ 2c^2\sigma_t^2(\beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1).
\end{aligned}
$$

6.3 Causality between drift μ_t and volatility η_t^2.

The causality between drift μ_t and volatility η_t^2, is equivalent to causality between the two following parametric linear combinations of r_t and σ_t^2:

$$
\begin{aligned}
\mu_t^* &= r_t(\beta_0 + c^*\gamma\delta_0) + \sigma_t^2(\beta_1 + c^*\gamma\delta_1), \\
\eta_t^{*2} &= r_t(\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0) + \sigma_t^2(\beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1) \\
&= \mu_t^* + c^{*2}\gamma^2(r_t\delta_0 + \sigma_t^2\delta_1).
\end{aligned}
$$

Conditional on information J_t, the drift and volatility satisfy a deterministic relationship iff $\beta_0\delta_1 - \beta_1\delta_0 = 0$. Otherwise, they satisfy a one-to-one relationship with r_t, σ_t^2, and we have :

$$
\begin{aligned}
r_t\beta_0 + \sigma_t^2\beta_1 &= \mu_t^* - \frac{1}{c^*\gamma}(\eta_t^{*2} - \mu_t^*), \\
r_t\delta_0 + \sigma_t^2\delta_1 &= \frac{1}{c^{*2}\gamma^2}(\eta_t^{*2} - \mu_t^*).
\end{aligned}
$$

(a) Nonlinear causality

The nonlinear causality analysis is based on the conditional log-Laplace transform Φ_t, say, of $\mu_{t+1}^*, \eta_{t+1}^{*2}$. We get :

$$
\begin{aligned}
\exp \Phi_t(u_1, u_2) &= E\left[\exp(-u_1\mu^*_{t+1} - u_2\eta^{*2}_{t+1})|J_t\right] \\
&= E\left[\exp\{-r_{t+1}\left[(u_1+u_2)(\beta_0 + c^*\gamma\delta_0) + u_2 c^{*2}\gamma^2\delta_0\right]\right. \\
&- \left.\sigma^2_{t+1}\left[(u_1+u_2)(\beta_1 + c^*\gamma\delta_1) + u_2 c^{*2}\gamma^2\delta_1\}|J_t\right]\right. .
\end{aligned}
$$

It follows that :

$$
\begin{aligned}
\Phi_t(u_1, u_2) &= \psi_t[(u_1+u_2)(\beta_0 + c^*\gamma\delta_0) + u_2 c^{*2}\gamma^2\delta_0, (u_1+u_2)(\beta_1 + c^*\gamma\delta_1) + u_2 c^{*2}\gamma^2\delta_1] \\
&= \psi_t[u_1\gamma_1 + u_2\gamma_1^*, u_1\gamma_2 + u_2\gamma_2^*],
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_1 &= \beta_0 + c^*\gamma\delta_0, \gamma_2 = \beta_1 + c^*\gamma\delta_1, \\
\gamma_1^* &= \beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0, \gamma_2^* = \beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1.
\end{aligned}
$$

Moreover, we have :

$$
\psi_t(v_1, v_2) = -\delta \log(1 + cv_1) - \delta^* \log\left[1 + c^*(v_2 + \frac{\gamma c v_1}{1 + cv_1})\right]
$$

$$
-\left[\mu^*_t - \frac{1}{c^*\gamma}(\eta^{*2}_t - \mu^*_t)\right]\frac{cv_1}{1 + cv_1}
$$

$$
-\frac{1}{c^*\gamma^2}(\eta^{*2}_t - \mu^*_t)\frac{v_2 + \dfrac{\gamma c v_1}{1 + cv_1}}{1 + c^*\left[v_2 + \dfrac{\gamma c v_1}{1 + cv_1}\right]}.
$$

These expressions allow us to find the noncausality conditions [see Appendix 3, v), vi), x)]:

Instantaneous nonlinear noncausality

The conditions are :

$$
(\beta_0 = \beta_1 = \gamma = 0) \text{ or } (\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0).
$$

Unidirectional nonlinear noncausality from volatility to drift

The conditions are :

$$\begin{aligned}
&\gamma_1 = \gamma_2 = 0 \\
&\Longleftrightarrow \beta_0 + c^*\gamma\delta_0 = \beta_1 + c^*\gamma\delta_1 = 0 \\
&\Longleftrightarrow \begin{cases} \beta_0 = \beta_1 = \delta_0 = \delta_1 = 0, \\ \\ \text{or } \beta_0 = \beta_1 = \gamma = 0. \end{cases}
\end{aligned}$$

Unidirectional nonlinear noncausality from drift to volatility

The conditions are :

$$\begin{aligned}
&\gamma_1^* = \gamma_2^* = 0 \\
&\Longleftrightarrow \beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0 = \beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1 = 0 \\
&\Longleftrightarrow \begin{cases} \beta_0 = \beta_1 = \delta_0 = \delta_1 = 0, \\ \\ \text{or } \ \beta_0 = \beta_1 = \gamma = 0. \end{cases}
\end{aligned}$$

(b) First-order causality

The following first-order conditional moments of drift and volatility are derived in Appendix 3 iii):

$$
\begin{aligned}
E(\mu^*_{t+1}|J_t) &= cte+ & c(\beta_0 + c^*\gamma\delta_0)\mu^*_t + \frac{1}{c^*\gamma^2}(\beta_1 + c^*\gamma\delta_1)(\eta^{*2}_t - \mu^*_t) \\
&= cte & +c\gamma_1\mu^*_t + \frac{1}{c^*\gamma^2}\gamma_2(\eta^{*2}_t - \mu^*_t); \\
E(\eta^{*2}_{t+1}|J_t) &= cte+ & c(\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0)\mu^*_t \\
&+ & \frac{1}{c^*\gamma^2}(\beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1)(\eta^{*2}_t - \mu^*_t) \\
&= cte & +c\gamma^*_1\mu^*_t + \frac{\gamma^*_2}{c^*\gamma^2}\gamma^2(\eta^{*2}_t - \mu^*_t).
\end{aligned}
$$

The conditions for first-order noncausality are the following:

Unidirectional first-order noncausality from volatility to drift:

$$
\begin{aligned}
&\gamma_2 = 0 \Longleftrightarrow \beta_1 + c^*\gamma\delta_1 = 0 \\
&\Longleftrightarrow (\beta_1 = \gamma = 0)\text{or } \ (\beta_1 = \delta_1 = 0).
\end{aligned}
$$

Unidirectional first-order noncausality from drift to volatility

$$
\begin{aligned}
&cc^*\gamma^2\gamma^*_1 - \gamma^*_2 = 0 \\
&\Longleftrightarrow (\gamma^*_1 = \gamma^*_2 = 0), \text{or } (\gamma^*_2 = \gamma = 0) \\
&\Longleftrightarrow (\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0) \text{ or } (\beta_1 = \gamma = 0).
\end{aligned}
$$

(c) Second-order causality

The conditions for second-order noncausality follow directly from the second-order expansion of the conditional log-Laplace transform (see Appendix 3 iv), viii), ix), x)].

Instantaneous second-order noncausality

The conditions are :

$$(\gamma_1 = \gamma_2 = 0) \text{ or } (\gamma_1 = \gamma_2^* = \gamma = 0) \text{ or } (\gamma_1^* = \gamma_2^* = 0) \text{ or } (\gamma_1^* = \gamma_2 = \gamma = 0)$$

$$\Longleftrightarrow (\beta_0 = \beta_1 = \gamma = 0) \text{ or } (\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0).$$

Unidirectional second-order noncausality from volatility to drift

The conditions are :

$$(\gamma_2 = \gamma = 0) \text{ or } (\gamma_2 = \gamma_1 = 0)$$

$$\Longleftrightarrow (\beta_1 = \gamma = 0) \text{ or } (\beta_1 = \delta_1 = 0).$$

Unidirectional second-order noncausality from drift to volatility

The conditions are :

$$(\gamma_2^* = \gamma = 0) \text{ or } (\gamma_2^* = \gamma_1^* = 0)$$

$$\Longleftrightarrow (\beta_1 = \gamma = 0) \text{ or } (\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0).$$

The noncausality conditions are summarized in Table 2 below.

Table 2: Noncausality Conditions

	drift $\rightarrow$ volat.	volat. $\rightarrow$ drift	instantaneous
first-order	$\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$ or $\beta_1 = \gamma = 0$	$\beta_1 = \delta_1 = 0$ or $\beta_1 = \gamma = 0$	
second-order	$\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$ or $\beta_1 = \gamma = 0$	$\beta_1 = \delta_1 = 0$ or $\beta_1 = \gamma = 0$	$\beta_0 = \beta_1 = \gamma = 0$ or $\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$
nonlinear	$\beta_0 = \beta_1 = \gamma = 0$ or $\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$	$\beta_0 = \beta_1 = \gamma = 0$ or $\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$	$\beta_0 = \beta_1 = \gamma = 0$ or $\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0$

The above table shows that the sets of noncausality restrictions involve the following elementary hypotheses :

$$
\begin{array}{ll}
H_1 = (\beta_1 = \gamma = 0) & \Rightarrow \text{ (second-order noncausality drift to volat)} \\
& + \text{ (second-order noncausality volat to drift)} \\
 & \\
H_2 = (\beta_0 = \beta_1 = \gamma = 0) & \Rightarrow \text{(nonlinear noncausality drift to volat)} \\
& +\text{(nonlinear noncausality volat to drift)} \\
& +\text{(nonlinear instantaneous noncausality)} \\
& = \text{(independence)} \\
 & \\
H_3 = (\beta_1 = \delta_1 = 0) & \Rightarrow \text{(second-order noncausality volat to drift)}
\end{array}
$$

The above parameter restrictions are easy to interpret. For instance, by considering the expressions of μ_t, η_t^2, given in Section 6.2, we see that both drift and volatility are constant, iff, $(\beta_0 = \beta_1 = \gamma = 0)$ or $(\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0)$. This explains the instantaneous independence revealed in the last column of Table 2.

7 Conclusion

This paper introduced a characterization of nonlinear causality based on the conditional Laplace transform. The noncausality hypotheses imply various restrictions on the parameters in models of volatility transmission, joint asset and derivative trading, and stochastic volatility examined in the paper. The examples show that the causality restrictions are often more complex in nonlinear dynamic models than in the standard Gaussian VAR model. Indeed, these hypotheses are in general written as unions of elementary hypotheses defined by equality constraints on the parameters, while in the Gaussian VAR they coincide with a unique elementary hypothesis. Therefore, when standard likelihood ratio tests are used for testing the causality hypotheses, the critical values have to be changed to account for the effect of a union of elementary hypotheses. Moreover, the asymptotic distributions of the likelihood ratio test statistics is not α-similar and can even be degenerate on the boundary of the null hypothesis of noncausality, especially for stochastic volatility models [see Gourieroux, Jasiak (2007)]. The development of causality tests is clearly out of the scope of this paper and left for future research.

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Appendix 1

Proof of Proposition 4

The conditions are easily derived by considering the first- and second-order derivatives of the conditional log-Laplace transform.

i) For example, the condition of instantaneous noncausality (Definition 1, i)) is equivalent to : $\dfrac{\partial^2 \psi_t(u_1, u_2)}{\partial u_2 \partial u_1'} = 0$. We get :

$$\begin{aligned} \frac{\partial \psi_t(u_1, u_2)}{\partial u_2} &= \frac{\partial \gamma_t}{\partial u_2}[\alpha_t(u_1) + u_2] \\ \frac{\partial^2 \psi_t(u_1, u_2)}{\partial u_2 \partial u_1'} &= \frac{\partial^2 \gamma_t}{\partial u_2 \partial u_2'}[\alpha_t(u_1) + u_2] \frac{\partial \alpha_t}{\partial u_1'}(u_1). \end{aligned}$$

The condition is satisfied iff, $\dfrac{\partial^2 \gamma_t}{\partial u_2 \partial u_2'}[\alpha_t(u_1) + u_2] \dfrac{\partial \alpha_t}{\partial u_1'}(u_1) = 0, \forall u_1, u_2,$

or, equivalently, iff $\dfrac{\partial^2 \gamma_t}{\partial u_2 \partial u_2'}(u_2) = 0, \forall u_2$, or $\dfrac{\partial \alpha_t}{\partial u_1}(u_1) = 0, \forall u_1$.

The result follows by observing that $\gamma_t(0) = 0, \alpha_t(0) = 0$.

ii) The condition of unidirectional second-order noncausality from x_2 to x_1 requires a detailed computation. Indeed, it involves the second-order derivative $\dfrac{\partial^2 \psi_t(u_1, u_2)}{\partial u_1 \partial u_1'}$. We get :

$$\begin{aligned} \frac{\partial \psi_t}{\partial u_1}(u_1, u_2) &= \frac{\partial \gamma_t}{\partial u_2'}[\alpha_t(u_1) + u_2] \frac{\partial \alpha_t(u_1)}{\partial u_1} + \frac{\partial \beta_t(u_1)}{\partial u_1} \\ &= \sum_{j=1}^{n_2} \frac{\partial \gamma_t}{\partial u_{2,j}}[\alpha_t(u_1) + u_2] \frac{\partial \alpha_{jt}}{\partial u_1}(u_1) + \frac{\partial \beta_t}{\partial u_1}(u_1). \end{aligned}$$

We find:

$$\frac{\partial^2\psi_t}{\partial u_1 \partial u_1'}(u_1,u_2) = \sum_{j=1}^{n_2} \frac{\partial \gamma_t}{\partial u_{2,j}}[\alpha_t(u_1)+u_2]\frac{\partial^2 \alpha_{j,t}}{\partial u_1 \partial u_1'}(u_1)$$

$$+ \sum_{j=1}^{n_2} \frac{\partial \alpha_{j,t}}{\partial u_1}(u_1)\frac{\partial^2 \gamma_t(\alpha_1(u_1)+u_2)}{\partial u_{2,j}\partial u_2'}\frac{\partial \alpha_t(u_1)}{\partial u_1'} + \frac{\partial^2 \beta_t(u_1)}{\partial u_1 \partial u_1'}.$$

The result follows for $u_1 = u_2 = 0$.

QED

Appendix 2

Noncausality from Covolatility to Volatilities

The conditional distribution of $(Y_{11,t+1}, Y_{22,t+1})$ is characterized by the conditional Laplace transform of the WAR process with $\gamma_{12} = 0$. We get:

$$\Psi_t(\gamma_{11},\gamma_{22})$$

$$= \frac{\exp Tr\left[M'\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}[Id - 2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}]^{-1}MY_t\right]}{\det\left[Id - 2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}\right]^{K/2}}.$$

Covolatility $Y_{12,t}$ does not nonlinearly cause $(Y_{11,t}, Y_{22,t})$, if and only if, this conditional Laplace transform does not depend on $Y_{12,t}$, or equivalently, if and only if,

$$Tr\left[M'\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}[Id - 2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}]^{-1}MY_t\right]$$

does not depend on $(Y_{12,t})$, for any admissible value of γ_{11}, γ_{22}.

Since this quantity is linear in Y_t, the above condition is also equivalent to:

$$Tr\left[M'\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}[Id-2\Sigma\begin{pmatrix}\gamma_{11} & 0\\ 0 & \gamma_{22}\end{pmatrix}]^{-1}M\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]=0,$$

or to the following condition (C1):

$$\begin{aligned}
&Tr\left[\begin{pmatrix}m_{11}\gamma_{11} & m_{21}\gamma_{22}\\ m_{12}\gamma_{11} & m_{22}\gamma_{22}\end{pmatrix}\begin{pmatrix}1-2\sigma_{22}\gamma_{22} & 2\sigma_{12}\gamma_{22}\\ 2\sigma_{12}\gamma_{11} & 1-2\sigma_{11}\gamma_{11}\end{pmatrix}\begin{pmatrix}m_{12} & m_{11}\\ m_{22} & m_{21}\end{pmatrix}\right]\\
= \;& (m_{11}\gamma_{11}, m_{21}\gamma_{22})\begin{pmatrix}1-2\sigma_{22}\gamma_{22} & 2\sigma_{12}\gamma_{22}\\ 2\sigma_{12}\gamma_{11} & 1-2\sigma_{11}\gamma_{11}\end{pmatrix}\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}\\
+ \;& (m_{12}\gamma_{11}, m_{22}\gamma_{22})\begin{pmatrix}1-2\sigma_{22}\gamma_{22} & 2\sigma_{12}\gamma_{22}\\ 2\sigma_{12}\gamma_{11} & 1-2\sigma_{11}\gamma_{11}\end{pmatrix}\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}=0,\ \forall\gamma_{11},\gamma_{22}.
\end{aligned}$$

This condition involves a polynomial of degree 2 in γ_{11}, γ_{22}, with a zero intercept. Let us first consider the term of degree 1 (corresponding to first-order causality). We get:

$$(m_{11}\gamma_{11}, m_{21}\gamma_{22})\begin{pmatrix}m_{12}\\ m_{22}\end{pmatrix}+(m_{12}\gamma_{11}, m_{22}\gamma_{22})\begin{pmatrix}m_{11}\\ m_{21}\end{pmatrix}=0,\ \forall\gamma_{11},\gamma_{22}.$$

This equality holds when: $m_{11}m_{12}=0$ and $m_{21}m_{22}=0$.

Therefore four cases can be considered:

Case 1: If $m_{11}=m_{21}=0$, we see that condition C1 is satisfied.

Case 2: If $m_{12}=m_{22}=0$, we see that condition C1 is satisfied.

Case 2: If $m_{12}=m_{21}=0$ and $m_{11}\neq 0, m_{22}\neq 0$, condition C1 becomes $\sigma_{12}m_{11}m_{22}=0$, that is, $\sigma_{12}=0$.

Case 3: If $m_{11}=m_{22}=0$ and $m_{12}\neq 0, m_{21}\neq 0$, condition C1 becomes $\sigma_{12}m_{12}m_{21}=0$, that is, $\sigma_{12}=0$.

Appendix 3

i) Joint conditional distribution of r_{t+1}, σ^2_{t+1}

We get:

$$\begin{aligned}
\exp \psi_t(u_1,u_2) &= E\left[\exp(-u_2\sigma^2_{t+1}) E\left[\exp(-u_1 r_{t+1}) | \underline{r_t}, \underline{\sigma^2_{t+1}}\right] | \underline{r_t}, \underline{\sigma^2_t}\right] \\
&= E\left[\exp(-u_2\sigma^2_{t+1}) \exp\left[-\delta \log(1+cu_1) - \frac{cu_1}{1+cu_1}(\beta_0 r_t + \beta_1 \sigma^2_t + \gamma\sigma^2_{t+1})\right] | \underline{r_t}, \underline{\sigma^2_t}\right] \\
&= \exp\left[-\delta \log(1+cu_1) - \frac{cu_1}{1+cu_1}(\beta_0 r_t + \beta_1 \sigma^2_t)\right] E\left[\exp\{-(u_2 + \frac{\gamma c u_1}{1+cu_1})\sigma^2_{t+1}\} | \underline{r_t}, \underline{\sigma^2_t}\right] \\
&= \exp\left[-\delta \log(1+cu_1) - \frac{cu_1}{1+cu_1}(\beta_0 r_t + \beta_1 \sigma^2_t) - \delta^* \log[1 + c^*(u_2 + \frac{\gamma c u_1}{1+cu_1})]\right. \\
&\quad \left. - \frac{c^*\left(u_2 + \frac{\gamma c u_1}{1+cu_1}\right)}{1 + c^*\left(u_2 + \frac{\gamma c u_1}{1+cu_1}\right)}(\delta_0 r_t + \delta_1 \sigma^2_t)\right].
\end{aligned}$$

ii) Conditional moments given J_t

We get :

$$\psi_t(u_1, 0)$$

$$= -\delta \log(1 + cu_1) - \delta^* \log(1 + cc^*\gamma \frac{u_1}{1 + cu_1})$$

$$-r_t \left\{ \beta_0 \frac{cu_1}{1 + cu_1} + \frac{cc^*\gamma\delta_0 \frac{u_1}{1 + cu_1}}{1 + cc^*\gamma \frac{u_1}{1 + cu_1}} \right\}$$

$$-\sigma_t^2 \left\{ \beta_1 \frac{cu_1}{1 + cu_1} + \frac{cc^*\gamma\delta_1 \frac{u_1}{1 + cu_1}}{1 + cc^*\gamma \frac{u_1}{1 + cu_1}} \right\}$$

$$\simeq -\delta \left[cu_1 - \frac{c^2 u_1^2}{2} \right] - \delta^* \left[cc^*\gamma u_1 - c^2 c^* \gamma u_1^2 - \frac{c^2 c^{*2} \gamma^2}{2} u_1^2 \right]$$

$$-r_t \left\{ \beta_0 (cu_1 - c^2 u_1^2) + cc^*\gamma\delta_0 u_1 (1 - cu_1 - cc^*\gamma u_1) \right\}$$

$$-\sigma_t^2 \left\{ \beta_1 (cu_1 - c^2 u_1^2) + cc^*\gamma\delta_1 u_1 (1 - cu_1 - cc^*\gamma u_1) \right\}.$$

It follows that:

$$\begin{aligned}
\mu_t &= E(r_{t+1} | \underline{r_t}, \underline{\sigma_t^2}) \\
&= c[\delta + c^*\gamma\delta^*] + cr_t[\beta_0 + c^*\gamma\delta_0] + c\sigma_t^2[\beta_1 + c^*\gamma\delta_1]; \\
\eta_t^2 &= V(r_{t+1} | \underline{r_t}, \underline{\sigma_t^2}) \\
&= c^2[\delta + 2c^*\gamma\delta^* + c^{*2}\gamma^2\delta^*] \\
&+ 2c^2 r_t[\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0] \\
&+ 2c^2\sigma_t^2[\beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1].
\end{aligned}$$

Finally, note that :

$$Cov\ (r_{t+1}, \sigma^2_{t+1} | J_t)$$

$$= Cov\ [E(r_{t+1} | \underline{r_t}, \underline{\sigma^2_{t+1}}) | J_t]$$

$$= c\gamma V(\sigma^2_{t+1} | J_t).$$

iii) First-order conditional moments of μ^*_t, η^{*2}_t.

We get :

$$\begin{aligned}
E(\mu^*_{t+1} | J_t) &= (\beta_0 + c^*\gamma\delta_0)\mu_t + (\beta_1 + c^*\gamma\delta_1)E(\sigma^2_{t+1} | J_t) \\
&= cte\ + c(\beta_0 + c^*\gamma\delta_0)\mu^*_t + \frac{1}{c^*\gamma^2}(\beta_1 + c^*\gamma\delta_1)(\eta^{*2}_t - \mu^*_t) \\
&= cte + c\gamma_1\mu^*_t + \frac{\gamma_2}{c^*\gamma^2}(\eta^{*2}_t - \mu^*_t); \\
E(\eta^{*2}_{t+1} | J_t) &= cte\ + c(\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0)\mu^*_t \\
&+ \frac{1}{c^*\gamma^2}(\beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1)(\eta^{*2}_t - \mu^*_t) \\
&= cte + c\gamma^*_1\mu^*_t + \frac{\gamma^*_2}{c^*\gamma^2}[\eta^{*2}_t - \mu^*_t].
\end{aligned}$$

iv) Second-order expansion of $\psi_t(v_1, v_2)$

We get :

$$\log(1 + cv_1) \simeq cv_1 - \frac{1}{2}c^2v_1^2,$$

$$cv_1/(1 + cv_1) \simeq cv_1 - c^2v_1^2,$$

$$\left(v_2 + \frac{\gamma cv_1}{1 + cv_1}\right)/[1 + c^*\left(v_2 + \frac{\gamma cv_1}{1 + cv_1}\right)] \simeq v_2 + \gamma cv_1 - \gamma c^2v_1^2 - c^*(v_2 + \gamma cv_1)^2,$$

$$\log[1 + c^*(v_2 + \frac{\gamma cv_1}{1 + cv_1})] \simeq c^*(v_2 + \gamma cv_1 - \gamma c^2v_1^2) - \frac{1}{2}c^{*2}(v_2 + \gamma cv_1)^2.$$

It follows that:

$$
\begin{aligned}
&\psi_t(v_1, v_2)\\
&\simeq -\delta c v_1 - \delta^* c^* (v_2 + \gamma c v_1)\\
&-[\mu_t^* - \frac{1}{c^*\gamma}(\eta_t^{*2} - \mu_t^*)] c v_1 - \frac{1}{c^*\gamma^2}(\eta_t^{*2} - \mu_t^*)(v_2 + \gamma c v_1)\\
&+\frac{1}{2}\delta c^2 v_1^2 + \delta^*[\gamma c^* c^2 v_1^2 + \frac{1}{2} c^{*2}(v_2 + \gamma c v_1)^2]\\
&+\mu_t^* c^2 v_1^2 + \frac{1}{\gamma^2}(\eta_t^{*2} - \mu_t^*)(v_2 + \gamma c v_1)^2.
\end{aligned}
$$

In particular, we get :

$$\Phi_t(u_1, 0) = \psi_t(u_1\gamma_1, u_1\gamma_2), \text{ with } \gamma_1 = \beta_0 + c^*\gamma\delta_0, \gamma_2 = \beta_1 + c^*\gamma\delta_1.$$

The second-order expansion of $\Phi_t(u_1, 0)$ is :

$$
\begin{aligned}
&\Phi_t(u_1, 0)\\
&\simeq -u_1\{\gamma_1 \delta c + \delta^* c^*(\gamma_2 + \gamma\gamma_1 c)\\
&+[\mu_t^* - \frac{1}{c^*\gamma}(\eta_t^{*2} - \mu_t^*)] c\gamma_1 + \frac{1}{c^*\gamma^2}(\eta_t^{*2} - \mu_t^*)(\gamma_2 + \gamma\gamma_1 c)\}\\
&+u_1^2\{\frac{1}{2}\delta c^2\gamma_1^2 + \delta^*[\gamma c^* c^2 \gamma_1^2 + \frac{1}{2}c^{*2}(\gamma_2 + \gamma\gamma_1 c)^2]\\
&+[\mu_t^* - \frac{1}{c^*\gamma}(\eta_t^{*2} - \mu_t^*)] c^2\gamma_1^2 + \frac{1}{c^*\gamma^2}(\eta_t^{*2} - \mu_t^*)[\gamma\gamma_1^2 c^2 + c^*(\gamma_2 + \gamma\gamma_1 c)^2]\}.
\end{aligned}
$$

Similarly, we get :

$$\Phi_t(0, u_2) = \psi_t(u_2\gamma_1^*, u_2\gamma_2^*),$$

$$\text{with : } \quad \gamma_1^* = \beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0, \gamma_2^* = \beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1.$$

The second-order expansion of $\Phi_t(0,u_2)$ is similar to the second-order expansion of $\Phi_t(u_1,0)$ after substituting u_2 for u_1, γ_1^* for γ_1 and γ_2^* for γ_2.

v) Unidirectional nonlinear noncausality from drift to volatility

The log-Laplace transform:

$$\Phi_t(0,u_2) = \psi_t(u_2\gamma_1^*, u_2\gamma_2^*), \text{ say},$$

does not depend on μ_t^*, if and only if,:

$$\left[1+\frac{1}{c^*\gamma}\right]\frac{c\gamma_1^*}{1+cu_2\gamma_1^*} - \frac{1}{c^*\gamma^2}\frac{\gamma_2^* + \dfrac{\gamma c\gamma_1^*}{1+c\gamma_1^*u_2}}{1+c^*u_2\left[\gamma_2^*+\dfrac{\gamma c\gamma_1^*}{1+c\gamma_1^*u_2}\right]} = 0, \forall u_2.$$

It is equivalent to :

$$\gamma_1^* = \gamma_2^* = 0,$$

or in terms of initial parameters :

$$\beta_0 + c^*\gamma\delta_0 + c^{*2}\gamma^2\delta_0 = \beta_1 + c^*\gamma\delta_1 + c^{*2}\gamma^2\delta_1 = 0.$$

vi) Unidirectional nonlinear noncausality from volatility to drift

We have :

$$\Phi_t(u_1,0) \quad = \quad \psi_t[u_1\gamma_1, u_1\gamma_2], \text{say}.$$

This conditional Laplace transform does not depend on η_t^{*2} if, and only if,

$$h(u_1) = \frac{c\gamma\gamma_1}{1+c\gamma_1u_1} - \frac{\gamma_2 + \dfrac{c\gamma\gamma_1}{1+c\gamma\gamma_1u_1}}{1+c^*u_1\left[\gamma_2+\dfrac{c\gamma\gamma_1}{1+c\gamma\gamma_1u_1}\right]} = 0, \forall u_1.$$

This condition is equivalent to : $\gamma_1 = \gamma_2 = 0$, or in terms of initial parameters : $\beta_0 + c^*\gamma\delta_0 = \beta_1 + c^*\gamma\delta_1 = 0$.

vii) Unidirectional second-order noncausality from volatility to drift

By considering the second-order expansion of $\Phi_t(u_1, 0)$ derived in Appendix iv), we get the two conditions :

$$\begin{cases} \gamma_2 = 0, \\ \\ c^*(\gamma_2 + \gamma\gamma_1 c)^2 = 0. \end{cases}$$

This system is equivalent to :

$$\gamma_2 = 0, \text{ and } c^*\gamma^2\gamma_1^2 c^2 = 0.$$

$$\iff (\gamma_2 = \gamma = 0) \text{ or } (\gamma_2 = \gamma_1 = 0)$$

viii) Unidirectional second-order noncausality from drift to volatility

By considering the second-order expansion of $\Phi_t(0, u_2)$ derived in Appendix 3 iv), and assuming that it does not depend on μ_t^*, we get the following two conditions :

$$\begin{cases} cc^*\gamma_1^*\gamma^2 - \gamma_2^* = 0, \\ \gamma_2^*[\gamma_2^* + 2\gamma\gamma_1^* c] = 0. \end{cases}$$

These conditions equivalent to :

$$(\gamma_2^* = \gamma_1^* = 0) \text{ or } (\gamma_2^* = \gamma = 0).$$

ix) Instantaneous second-order noncausality

The noncausality conditions are obtained by setting the coefficient of the cross-term $u_1 u_2$ in the second-order expansion of $\Phi_t(u_1, u_2)$ equal to zero. The quadratic term in the expansion of $\psi_t(v_1, v_2)$ is [see Appendix 3 iv)] :

$$v_1^2[\frac{1}{2}\delta c^2 + \delta^*\gamma c^* c^2 + c^2\mu_t^*]$$

$$+(v_2 + \gamma c v_1)^2[\frac{1}{2}\delta^* c^{*2} + \frac{1}{\gamma^2}(\eta_t^{*2} - \mu_t^*)].$$

The coefficient of the cross-term in $\Phi_t(u_1, u_2)$:

$$\gamma_1\gamma_1^*[\frac{1}{2}\delta c^2 + \delta^*\gamma c^* c^2 + c^2\mu_t^*]$$

$$+(\gamma_2 + \gamma\gamma_1 c)(\gamma_2^* + \gamma\gamma_1^* c)[\frac{1}{2}\delta^* c^{*2} + \frac{1}{\gamma^2}(\eta_t^{*2} - \mu_t^*)],$$

is equal to zero for any admissible value of μ_t^*, η_t^{*2} iff, $\gamma_1\gamma_1^* = 0$ and $(\gamma_2 + \gamma\gamma_1 c)(\gamma_2^* + \gamma\gamma_1^* c) = 0$.

This condition is equivalent to :

$$(\gamma_1 = \gamma_2 = 0) \text{ or } (\gamma_1 = \gamma_2^* = \gamma = 0) \text{ or } (\gamma_1^* = \gamma_2^* = 0) \text{ or } (\gamma_1^* = \gamma_2 = \gamma = 0).$$

This condition can be written in terms of the primitive parameters as :

$$(\beta_0 = \beta_1 = \gamma = 0) \text{ or} (\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0).$$

x) Instantaneous noncausality

It is known that this noncausality condition requires the corresponding second-order condition to be satisfied. Thus, we can consider the model constrained by either $(\beta_0 = \beta_1 = \gamma = 0)$, or $(\beta_0 = \beta_1 = \delta_0 = \delta_1 = 0)$. In both cases, we get :

$$\Phi_t(u_1, u_2) = \psi_t(0, 0) = 0.$$

Thus, the instantaneous noncausality condition is satisfied.