

# Revisiting Identification and Estimation in Structural VARMA Models

C., GOURIEROUX <sup>(1)</sup> and A., MONFORT <sup>(2)</sup>

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<sup>1</sup>CREST and University of Toronto.

<sup>2</sup>CREST and Banque de France.

# Revisiting Identification in Structural VARMA Models

## Abstract

The basic assumption of a structural VARMA model (SVARMA) is that it is driven by a white noise whose components are uncorrelated (or independent) and are interpreted as economic shocks, called "structural" shocks. These models have to face two kinds of identification problems. The first identification problem is "static" and is due to the fact that there is an infinite number of linear transformations of a given random vector making its components uncorrelated. The second identification problem is "dynamic" and is a consequence of the fact that the SVARMA process may have a non invertible AR and/or a MA matrix polynomial but, still, has the same second order properties as a VARMA process in which both the AR and MA matrix polynomials are invertible (the fundamental representation). Moreover the standard Box-Jenkins approach automatically estimates the fundamental representation and, therefore, may lead to misspecified Impulse Response Functions.

The aim of this paper is to explain that these difficulties are mainly due to the Gaussian assumption underlying the BJ approach, and that both identification challenges are solved in a non Gaussian framework. We also develop simple new parametric and semi-parametric estimation methods when there is nonfundamentalness in either the moving-average, or the autoregressive dynamics.

**Keywords :** Structural VAR, Fundamental Representation, Noncausal Process, Shock, Impulse Response Function, Incomplete Maximum Likelihood, Pseudo-Maximum Likelihood.

# 1 Introduction

The basic assumption of a structural VARMA model (SVARMA) is that it is driven by a white noise whose components are uncorrelated (or independent) and are interpreted as economic shocks<sup>3</sup>, called "structural" shocks. These models have to face two kinds of identification problems.

First the components of the white noise appearing in the reduced form VARMA are instantaneously correlated and the shock vector must be derived from this white noise by a linear transformation eliminating these instantaneous correlations. The snag is that this can be done in an infinite number of ways and there is a huge literature trying to solve this "static" identification problem by adding restrictions on the short run impact of a shock [see e.g. Bernanke (1980), Sims (1986), Rubio-Ramirez, Waggoner, Zha (2010)], or on the long run impact [see e.g. Blanchard, Quah (1989), Faust, Leeper (1997), Erceg, Guerrieri, Gust (2005), Christiano, Eichenbaum, Vigfusson (2006)], as well as on the sign of some impulse response functions [see e.g. Uhlig (2005), Chari, Kehoe, McGrattan (2008), Mountford, Uhlig (2009)].

A second identification problem comes from the fact that the SVARMA may feature a non-invertible autoregressive (AR) or, more often, a non-invertible moving average (MA) matrix polynomial. This type of situation may occur, for instance, when the SVARMA is deduced from business cycle models [see e.g. Kydland, Prescott (1982), Francis, Ramey (2005), Gali, Rabanal (2005)], or from log-linear approximations of Dynamic Stochastic General Equilibrium (DSGE) models involving rational expectations [see e.g. Hansen, Sargent (1991), Smet, Wouters (2003), Christiano, Eichenbaum, Vigfussen (2007), Leeper, Walker, Yang (2013)]. Typically the matrix MA polynomial is not invertible and the shock vector is not simply linearly linked to the innovation of the process [see Lippi, Reichlin (1993) (1994)]. Moreover the SVARMA process has exactly the same second order properties as another VARMA process which has an invertible MA part (the fundamental representation) and, in the Gaussian case, both processes are observationally equivalent. This creates an identification problem (a "dynamic" identification problem), which is exacerbated by the fact that the standard Box-Jenkins approach (i.e. the Gaussian Pseudo Maximum Likelihood method based on a VAR approximation of the VARMA process obtained by inverting

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<sup>3</sup>Our paper will not consider the debate about how structural are the parameters and the shocks in SVARMA models [see e.g. Pesaran, Smith (2011) and the reference therein].

the MA part) provides a consistent estimation of the fundamental representation and, therefore, may lead to misspecified Impulse Response Functions.

The aim of this paper is to explain that these difficulties are due to the Gaussian assumption underlying the BJ approach, and that the identification issues are solved in a non Gaussian framework. We also introduce simple estimation approaches when there is nonfundamentalness either in the moving-average, or in the autoregressive dynamics.

In Section 2, we consider a vector autoregressive moving average process, with roots of the autoregressive and/or moving average polynomials that are not necessarily outside the unit circle. We focus on the two-sided moving average and autoregressive representations and on the fact that the economic shocks are not necessarily interpretable in terms of causal innovations<sup>4</sup>. We review the different types of nonfundamental representations in the moving average dynamics given in the literature. We also explain that nonfundamentalness can arise in the autoregressive dynamics, when rational expectations are included in the model. In particular we show that there always exists an infinite number of stationary solutions in a rational expectation model, if we do not impose these solutions to have a finite variance. Next we discuss the identification issue in the Gaussian case and explain why the standard BJ approach based on Gaussian pseudo-likelihood suffers from these identification issues.

Section 3 is the core of the paper. We consider the case of non Gaussian observable SVARMA process, based on recent literature on the so-called non-causal processes [see e.g. Brockwell, Davis (1991), Rosenblatt (2000) for an introduction]. We explain that all standard identification problems encountered in the SVARMA analysis disappear when the shocks are not Gaussian. In Section 4 we suggest new estimation methods to improve the standard SVAR methodology. We first consider a parametric SVARMA with nonfundamental representation in the moving average dynamics only and introduce new Incomplete Maximum Likelihood (IML) and Simulated Pseudo Maximum Likelihood (SPML) approaches. Then we introduce a semi-parametric approach based on instrumental variables to estimate a SVARMA model with nonfundamental representation. Finally, we explain how to simulate the path of a SVAR process with nonfundamental representation, which is required to analyze the finite sample properties of the parametric or semi-parametric estimators by bootstrap. Section 5 concludes. Some complements are gathered

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<sup>4</sup>See Appendix 1 for precise definitions of the different notions of innovations.

in appendices.

## 2 Dynamic Linear Model and Nonfundamentality

### 2.1 The dynamic model

Despite the standard Vector Autoregressive (VAR) terminology, the linear dynamic models deduced from structural models have in general both autoregressive and moving average dynamics. The VARMA model is the following :

$$\Phi(L)Y_t = \Theta(L)\varepsilon_t, \quad (2.1)$$

where  $Y_t$  is the  $n$ -dimensional vector of observations at date  $t$ ,  $\varepsilon_t$  is the  $m$ -dimensional vector of shocks,

$$\Phi(L) = Id - \Phi_1 L - \dots - \Phi_p L^p, \Theta(L) = Id - \Theta_1 L - \dots - \Theta_q L^q, \quad (2.2)$$

and the matrix autoregressive and moving average lag-polynomials are of degree  $p$  and  $q$ , respectively. The underlying structural model can include state variables, which are not necessarily observable. This explains why the number of shocks  $m$ , corresponding to the number of state variables can be larger than the number of observed variables  $Y$ , even if models considered in practice<sup>5</sup> are often such that  $n = m$ .

Let us now introduce the following assumptions on model (2.1) :

**a.1 : Assumption on dimensions**  $n = m$ .

Two other assumptions are also made :

**a.2 : Assumption on shocks**

i) The shocks  $\varepsilon_t$  are independently and identically distributed. They have some moments :  $\exists s > 0, E(\|\varepsilon_t\|^s) < \infty$ .

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<sup>5</sup>See e.g. Hansen, Sargent (1991), p83, Lippi, Reichlin (1994), Giannone, Reichlin (2006), p457, Fernandez-Villaverde et al. (2007) Section C, for this assumption.

ii) They can be written as  $\varepsilon_t = C\eta_t \Leftrightarrow \eta_t = C^{-1}\varepsilon_t$ , where the components of  $\eta_t$  are mutually independent. The components of  $\eta_t$  are the structural shocks.

Assumption a.2 i) on the shocks is standard in the literature. For instance for  $s = 2$ , the existence of second-order moments underlies the Box-Jenkins methodology. We extend the set of possible shocks to allow for shocks with infinite variance, or even no mean, in line with the financial literature. This will also allow for considering stationary solutions of rational expectation models containing explosive speculative bubbles (see Section 2.2). Assumption a.2 ii) is needed for allowing separate shocks on the system when defining the impulse response functions.

### **a.3 : Assumption of left coprimeness on the lag-polynomials**

If  $\Phi(L)$  and  $\Theta(L)$  have a left common factor  $C(L)$ , say, such that :  $\Phi(L) = C(L)\tilde{\Phi}(L)$ ,  $\Theta(L) = C(L)\tilde{\Theta}(L)$ , then  $\det C(L)$  is independent of  $L$ .

This condition ensures that the VARMA representation is minimal in the sense that all possible simplifications have been already done [see Hannan, Deistler (1988) Chap 2 for more details] This condition will greatly simplify the discussion in the next sections. It is often forgotten in structural setting and it might be necessary to test for the minimality of the representation. This is clearly out of the scope of this paper.<sup>6</sup>

### **a.4 : Assumption on the observable process**

The process  $(Y_t)$  is strongly stationary.

The existence of a stationary solution to system (2.1)-(2.2) is ensured under the next assumption.

### **a.5 : Assumption on the dynamics**

All the roots of  $\det \Phi(L)$  have a modulus different from 1.

Under Assumptions a.1 - a.5, linear dynamic system (2.1)-(2.2) has a unique strictly stationary solution, such that  $E\|Y_t\|^s < \infty$  [see e.g. the

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<sup>6</sup>See however Deistler, Schrader (1979) for a study of identifiability without coprimeness, and Gourieroux, Monfort, Renault (1989) for the test of coprimeness, i.e. common roots, for one-dimensional ARMA processes.

discussion in Gourieroux, Zakoian (2014) a,b].

When all the roots of  $\det \Phi(z)$  lie outside the unit circle, it is easy to derive the inverse of the polynomial operator  $\Phi(L)$  as a convergent one-sided series in the lag operator  $L$  :

$$\begin{aligned} \Phi(L)Y_t &= \Theta(L)\varepsilon_t \\ \iff Y_t &= \Phi(L)^{-1}\Theta(L)\varepsilon_t \equiv \sum_{j=0}^{\infty} A_j L^j \varepsilon_t = \sum_{j=0}^{\infty} A_j \varepsilon_{t-j}. \end{aligned} \quad (2.3)$$

Similarly when all the roots of  $\det \Theta(z)$  lie outside the unit circle,  $Y_t$  has a one-sided autoregressive representation :

$$\Theta^{-1}(L)\Phi(L)Y_t \equiv \sum_{j=0}^{\infty} B_j L^j Y_t = \varepsilon_t.$$

From the macroeconomic literature we know that SVARMA models do not always have roots of the autoregressive or moving average located outside the unit circle (see Section 2.2). In that case, it is still possible to invert the autoregressive polynomial operator to get a two-sided moving average representation as the stationary solution :

$$Y_t = \sum_{j=-\infty}^{+\infty} A_j \varepsilon_{t-j}. \quad (2.4)$$

Similarly if  $\det \Theta(z)$  has no roots on the unit circle we get a two-sided autoregressive representation :

$$\sum_{j=-\infty}^{\infty} B_j Y_{t-j} = \varepsilon_t.$$

**Example :** The approximation of a DSGE model is often presented in a state space form including the dynamics of both the observable and state variables. For instance a typical specification is [see e.g. Fernandez-Villaverde et al. (2007)] :

$$\begin{cases} X_t = AX_{t-1} + B\varepsilon_t, \\ Y_t = CX_{t-1} + D\varepsilon_t, \end{cases} \quad (2.5)$$

where  $X_t, Y_t$  denote the values of the state and observable variables, respectively, and  $D$  is assumed invertible. It is easy to derive a VARMA representation of  $Y_t$  from (2.5). We have :

$$\left( Id - \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} L \right) \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix} \varepsilon_t,$$

or equivalently :

$$\det \psi(L) \begin{pmatrix} X_t \\ Y_t \end{pmatrix} = \psi^*(L) \begin{pmatrix} B \\ D \end{pmatrix} \varepsilon_t,$$

where  $\psi^*(L)$  is the adjoint matrix of  $\psi(L) = Id - \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} L$ . From which we deduce :

$$\det \psi(L) Y_t = (\psi_{21}^*(L) B + \psi_{22}^*(L) D) \varepsilon_t.$$

We get a VARMA representation of the process of observable values. In such a state space representation, we might have ex-ante roots of  $\det \psi(z)$  inside the unit circle, i.e. "ill located". However some difficulties can also arise if these roots are all well-located. Indeed, passing from the VAR state space form for  $(X_t, Y_t)$  to the VARMA specification for  $Y_t$  alone, we might exhibit ill-located roots in the MA component whereas all roots of the AR component are well-located <sup>7</sup> (see the discussion in Section 2.2).

Let us study the consequences of ill-located roots of  $\det \Theta(z)$ . For expository purpose, we consider a one-dimensional ARMA (1,1) process :

$$(1 - \varphi L) y_t = (1 - \theta L) \varepsilon_t, \quad (2.6)$$

where  $|\varphi| < 1$  and  $|\theta| > 1$ . Thus the root of  $\det \Theta(z)$  is inside the unit circle. To get the (infinite) pure autoregressive representation of process  $y_t$ , we have to invert  $(1 - \theta L)$ . We get :

$$\begin{aligned} (1 - \varphi L) y_t &= \left(1 - \frac{1}{\theta} L^{-1}\right) (-\theta L \varepsilon_t) \\ \Leftrightarrow \left(1 - \frac{1}{\theta} L^{-1}\right)^{-1} (1 - \varphi L) y_t &= -\theta L \varepsilon_t. \end{aligned} \quad (2.7)$$

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<sup>7</sup>Note however that this VARMA representation is not necessarily minimal, i.e. satisfying the left coprimeness assumption.



This formula reveals that :

- i) the process has a two-sided autoregressive representation;
- ii) the error appropriate for a mathematical analysis of the dynamics is rather the time shifted process  $\tilde{\varepsilon}_t = -\theta L\varepsilon_t = -\theta\varepsilon_{t-1}$  [see e.g. Lanne, Saikkonen (2013), Gouriéroux, Jasiak (2014)], than  $\varepsilon_t$  itself. Equation (2.6) becomes :

$$y_t - \varphi y_{t-1} = -\frac{1}{\theta}\tilde{\varepsilon}_{t+1} + \tilde{\varepsilon}_t, \text{ with a future value of process } \tilde{\varepsilon} \text{ on the right hand side.}$$

- iii) The error term  $\varepsilon_t$  is not the "causal innovation" of process  $y_t$  (see appendix 1), i.e.

$$\varepsilon_t \neq y_t - E[y_t|y_{t-1}, y_{t-2}, \dots], \text{ if this conditional expectation exists.}$$

To summarize, under assumption a.3, the error term in the VARMA representation is equal to the causal innovation of the process if the roots of  $\det \Phi(z)$  and  $\det \Theta(z)$  are all outside the unit circle. Under this condition, we say that process  $Y_t$  has a fundamental VARMA representation [see e.g. Hansen, Sargent (1980), p18, (1991), p79, and Lippi, Reichlin (1994) for the introduction of this terminology in the macroeconomic literature].<sup>8</sup> Otherwise,  $\varepsilon_t$  is not equal to the causal innovation and future values are involved in either the pure moving average, or the pure autoregressive representation of the process. It is a nonfundamental VARMA representation. As noted in Lanne, Saikkonen (2011) "nonfundamental solutions have typically been represented by noninvertible moving average models. However, noncausal autoregressive and noninvertible moving average models closely approximate each other".<sup>9</sup> Thus the nonfundamentalness has to be considered in a more symmetric way in both the AR and MA polynomials defining the dynamics.

The macroeconomic specific terminology fundamental vs nonfundamental representation must be related to other standard time series terminologies. In time series a (linear) process is causal (resp. invertible), if it

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<sup>8</sup>The term "fundamental" is likely due to Kolmogorov and appears in Rozanov (1960), p367, (1967), p56, to define the "fundamental process", that is the second-order white noise process involved in the Wold decomposition of a weak stationary process. At any date  $t$ , the information contained in the current and past values of the fundamental process coincides with the information contained in the current and past values of the observations.

<sup>9</sup>Indeed, when  $|\theta| > 1$ , the truncated version of  $[1 - 1/\theta L^{-1}]^{-1}$  is  $1 + 1/\theta L^{-1} + \dots + (1/\theta)^p L^{-p}$  with also all roots in  $L$  inside the unit circle, with the same modulus  $1/|\theta|$ .

admits a one-sided moving average representation (resp. a one-sided autoregressive representation). For a VARMA model with  $n = m$ , satisfying the left coprimeness assumption, the process is causal (resp. invertible) if  $\det \Phi(z)$  (resp.  $\det \Theta(z)$ ) has all its roots outside the unit circle. Thus the VARMA representation is fundamental if it is both causal and invertible. <sup>10</sup>

## 2.2 Nonfundamentality

There exist different sources of nonfundamentality in SVAR models [see the discussion in Alessi, Barigozzi, Capasso (2011)].

i) Some are due to the dynamics of exogenous variables in the system. A well-known example appears in the comment of the Blanchard, Quah model [Blanchard, Quah (1989)] by Lippi, Reichlin (1993). The productivity,  $x_t$ , can be written as :

$$x_t = u_t + \theta u_{t-1},$$

where  $u_t$  denotes the shock on productivity. It may be realistic to assume that the impact of the shock is not instantaneous and is maximal with a lag, i.e. that  $\theta > 1$ . This is the learning-by-doing hypothesis.

ii) As mentioned in Section 2.1., nonfundamentality can also arise from a lack of observability. Fernandez-Villaverde et al. (2007) give the example of a state space representation of the surplus in a permanent income consumption model [see Lof (2013), Section 3, for another example]. The state space model is of the following type :

$$\begin{cases} x_t = ax_{t-1} + (1 - 1/R)w_t, 0 < a < 1, \\ y_t = -x_{t-1} + 1/Rw_t, \end{cases}$$

where  $x_t$  (resp.  $y_t$ ) denotes the consumption (resp. the surplus),  $R > 1$  a constant gross interest rate on financial assets, and  $w_{t+1}$  is an i.i.d. labor income process. From the first equation, we deduce :

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<sup>10</sup>The terminology fundamental can be misleading, in particular since fundamental shock and structural shock are considered as equivalent notions (see e.g. the description of the scientific works of Nobel prizes Sargent and Sims in Economic Sciences Prize Committee (2011), or Evans, Marshall (2005)). Moreover a fundamental shock is also defined as the non bubble component of a structural shock, i.e. the component with a permanent effect on the economy in Velinov, Chen (2013).

$$x_t = \frac{(1 - 1/R)}{1 - aL} w_t,$$

and by substituting in the second equation, we get the dynamics of  $y_t$  :

$$y_t = \left[1/R - \frac{L(1 - 1/R)}{1 - aL}\right] w_t = \frac{1/R + L(\frac{1-a}{R} - 1)}{1 - aL} w_t.$$

Thus the root of the moving-average lag-polynomial is equal to  $[R - (1 - a)]^{-1}$ , and smaller than one when  $a$  is sufficiently close to one. <sup>11</sup>

iii) Other sources of nonfundamentalness are the rational expectations introduced in such models. The simple example of Hansen, Sargent (1991) shows that the rational expectation can create a nonfundamental moving average dynamics. If the economic variable  $y_t$ , is defined as :

$$y_t = E_t\left(\sum_{h=0}^{\infty} \beta^h w_{t+h}\right), \text{ with } w_t = u_t - \theta u_{t-1}, 0 < \beta < 1, |\theta| < 1.$$

and if the information set available at date  $t$  is  $I_t = (u_t, u_{t-1}, \dots)$ , we get :

$$y_t = (1 - \beta\theta)u_t - \theta u_{t-1}.$$

The root of the moving average polynomial is  $(1 - \beta\theta)/\theta$ . This root can be larger or smaller than 1, depending on the values of  $\beta$  and  $\theta$ .

iv) Nonfundamentalness may also occur when the economic agent and econometrician information sets are not aligned. The literature on information flows applied, for instance, to fiscal foresight or productivity belongs to this category [see e.g. Feve, Matheron, Sahuc (2009), Feve, Jihoud (2010), Forni, Gambetti (2010), Leeper, Walker, Yang (2013)]. A stylized model is [see Feve, Matheron, Sahuc (2009)] :

$$y_t = aE_t y_{t+1} + x_t,$$

$$x_t = \varepsilon_{t-q},$$

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<sup>11</sup>This reasoning does not hold for  $a = 1$ , which was precisely the case considered in Fernandez-Villaverde et al. (2007), where  $x_t$  and  $y_t$  are nonstationary co-integrated processes. Indeed their equation (5) assumes the stationarity of the  $y$  process and is not compatible with the assumption of cointegrated model.

where  $\varepsilon_t$  is a white noise.

If  $|a| < 1$  the forward solution is easily seen to be :

$$y_t = \sum_{i=0}^q a^{q-i} \varepsilon_{t-i}. \quad (2.8)$$

The roots of  $\Theta(L) = a^q \sum_{i=0}^q a^{-i} L^i = a^q \frac{1 - (a^{-1}L)^{q+1}}{1 - a^{-1}L}$  are  $a \exp(2ik\pi/(q+1))$ ,  $k = 1, \dots, q$ , whose modulus is  $|a| < 1$ . Therefore  $\Theta(L)$  is noninvertible and the MA is nonfundamental.

v) When the variable of interest can be interpreted as a prediction error, nonfundamental representation may also appear [see Hansen, Hodrick (1980)]. For instance if  $y_t$  is the price of an asset at  $t$ ,  $E_{t-2}y_t$  can be interpreted as the futures price at  $t-2$  (if the agents are risk-neutral) and, also, the forward price (if, moreover, the interest rates are exogenous). The spread between the spot price and the futures price is  $s_t = y_t - E_{t-2}y_t$  and if  $y_t$  is, for instance, an invertible MA(2) process  $y_t = \varepsilon_t + \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} = \Theta(L)\varepsilon_t$  we get  $s_t = \varepsilon_t + \theta_1\varepsilon_{t-1} = \Theta_1(L)\varepsilon_t$  which is not necessarily invertible. For example if  $\Theta(L) = (1 - \theta L)^2$  with  $|\theta| < 1$ , we have  $\Theta_1(L) = 1 - 2\theta L$ , which is not invertible as soon as  $|\theta| > \frac{1}{2}$ .

vi) The presence of rational expectations can also imply ill-located roots in the autoregressive dynamics. Let us consider again the univariate linear expectation model for prices à la Taylor (1977), Diba, Grossmann (1988) :

$$y_t = aE_t(y_{t+1}) + z_t, a > 0, \quad (2.9)$$

where  $z_t$  is an exogenous variable.

Since the equilibrium equation is obtained by matching the demand and supply, the information set as well as the endogenous and exogenous variables  $y_t, z_t$  depend on the structural shocks to the demand and supply. Therefore there are two underlying shocks  $\varepsilon_t$  and  $w_t$ , say. Let us assume that these shocks are independent and square integrable. For expository purpose we assume that  $z_t = \varepsilon_t$  is a strong white noise. It is well known that the set of square integrable stationary equilibrium prices depends on coefficient  $a$  [see e.g. Blanchard (1978), Gourieroux, Laffont, Monfort (1982)]. If  $a < 1$ , there is a unique solution  $y_t^0 = \varepsilon_t$ . If  $a > 1$ , there is an infinite number of solutions

obtained by considering the convex combinations of the forward solution  $y_t^0$  and of the perfect foresight or backward solution :

$$y_t^1 = ay_{t+1}^1 + \varepsilon_t \iff y_t^1 = \frac{L}{L-a}\varepsilon_t.$$

In fact **this dynamic rational expectation model has many more stationary solutions, if we do not restrict the equilibrium price to be square integrable.** In order to reveal other solutions, let us first recall the properties of the noncausal stable AR(1) process [see Gouriou, Zakoian (2014), a].

A noncausal stable AR(1) process is a strongly stationary process satisfying the noncausal autoregression :

$$y_t^* = \rho y_{t+1}^* + \varepsilon_t^*, |\rho| < 1, \quad (2.10)$$

where the  $\varepsilon_t^*$  are i.i.d. variables with a stable distribution with stability index  $s, 0 < s < 1$ . This process is also a Markov process in calendar time and admits a nonlinear autoregressive representation [see e.g. Rosenblatt (2000)] :

$$y_t^* = g(y_{t-1}^*, \eta_t; s, \rho), \quad (2.11)$$

where  $\eta_t$  is a strong Gaussian white noise.

Due to the fat tails of  $\varepsilon_t^*$ , the process  $y_t^*$  admits neither first, nor second-order unconditional finite moments, nor first and second-order moments conditional to the future. Nevertheless, it is shown in Gouriou, Zakoian (2014)a that this process has a first-order conditional moment given the past and that this conditional expectation is given by :

$$E(y_{t+1}^* | \underline{y}_t^*) = |\rho|^{(s-1)} y_t^*, \quad (2.12)$$

where the autoregressive coefficient  $|\rho|^{(s-1)}$  is larger than 1.

Let us now consider again rational expectation model (2.9), assume that the additional shock  $w_t$  is standard Gaussian and consider the noncausal stable AR(1) process defined by :

$$y_t^*(s, \rho) = g[y_{t-1}^*(s, \rho), w_t; s, \rho] \quad (2.13)$$

$$\Leftrightarrow y_t^*(s, \rho) = \rho y_{t+1}^*(s, \rho) + \varepsilon_t^*(s, \rho). \quad (2.14)$$

We see that the process :

$$y_t(s, \rho) = y_t^*(s, \rho) + \varepsilon_t, \quad (2.15)$$

is a solution to dynamic rational expectation model (2.9), whenever

$$|\rho| = a^{1/(1-s)}, \quad (2.16)$$

since

$$\begin{aligned} E_t(y_{t+1}) &= E_t y_{t+1}^* \\ &= |\rho|^{s-1} y_t^* = \frac{1}{a} y_t^* \end{aligned}$$

and  $y_t^* = a E_t y_{t+1}$ .

This implies that, even in the case  $a < 1$ , the RE model (2.9) has an infinite number of stationary solutions. Indeed, the stability index  $s$  can be chosen arbitrarily and by taking linear combinations, we find that any process of the type :

$$y_t = \sum_s \lambda(s) y_t^*[s, a^{1/(1-s)}] + \varepsilon_t,$$

is a stationary solution <sup>12</sup>, since  $E_t y_{t+1} = \frac{1}{a} \sum_s \lambda(s) y_t^*[s, a^{1/(1-s)}]$ .

Equation (2.15) implies :

$$\begin{aligned} y_t(s, \rho) &= \frac{\varepsilon_t^*(s, \rho)}{1 - \rho L^{-1}} + \varepsilon_t \\ \Leftrightarrow (L - \rho) y_t(s, \rho) &= L \varepsilon_t^*(s, \rho) + (L - \rho) \varepsilon_t. \end{aligned} \quad (2.17)$$

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<sup>12</sup>It is usual to select the forward solution as if it were the unique solution of the rational expectation model in the case  $a < 1$ . This practice neglects all stationary solutions with speculative bubbles [see e.g. Pesaran, Smith (2011), p7, Leeper et al. (2013), eq. (4) and eq. (17) for recent examples of this practice].

and the ARMA representation of  $(y_t, z_t)$  with  $z_t = \varepsilon_t$ , that is the system :

$$\begin{cases} (L - \rho)y_t(s, \rho) - (L - \rho) z_t = L\varepsilon_t^*(s, \rho), \\ z_t = \varepsilon_t, \end{cases}$$

is such that one root of the autoregressive polynomial is inside the unit circle. Such solutions are interesting since  $y_t^*(s, \rho)$  is a stationary speculative bubble component [see Gourieroux, Zakoian (2014) a].

Finally the standard Box-Jenkins approach applied to the bivariate series  $[y_t(s, \rho), z_t]$  is unable to find the underlying shocks  $\varepsilon_t^*(s, \rho)$ . First the BJ method assumes the square integrability of  $y_t(s, \rho)$  whereas this process has no mean. Second the BJ approach is a linear approach, which cannot accomodate the nonlinear innovation  $w_t$  in (2.13). However, as seen in Section 3, if both process  $y_t$  and  $z_t$  are observable, it is possible to identify the errors  $(\varepsilon_t, \varepsilon_t^*(s, \rho))$  of the nonfundamental VARMA representation of  $y_t$ , and by using the link between the causal and noncausal representations of  $y_t$  to recover the economic shocks  $(\varepsilon_t, w_t)$ . Thus, contrary to a common belief [see e.g. Lanne, Saikkonen (2011), p1], "the presence of noncausality" (does not) necessarily indicate that the agents are able to forecast a part of the future values of the economic variable in question by information unknown to econometrician". Indeed we have to focus on economic shocks  $(\varepsilon_t, w_t)$ , and the shocks  $[\varepsilon_t^*(s, \rho), \varepsilon_t]$ , are just convenient for estimation purpose. In particular the impulse response functions<sup>13</sup> have to be derived by applying shocks to  $(\varepsilon_t, w_t)$ , and not to nonfundamental forecast errors  $[\varepsilon_t^*(s, \rho), \varepsilon_t]$ . The multipliers are not derived from the nonfundamental VARMA representation, but from the associated nonlinear causal autoregressive representation [see Koop, Pesaran, Potter (1996), Gourieroux, Jasiak (2005), for the definition of impulse response functions in a general framework].

### 2.3 The limits of the Gaussian approach

The Box-Jenkins methodology is based on the assumption of Gaussian errors. Let us first discuss the identification issue for a Gaussian VAR, say, with  $n = m$ . The results are similar for a Gaussian VARMA. The observable process satisfies the autogressive dynamics :

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<sup>13</sup>The impulse response analysis describes how structural shocks propagate through the macroeconomy.

$$\Phi(L)Y_t = C\eta_t, \eta_t \sim IIN(0, Id), \quad (2.18)$$

where  $\Phi(L) = Id - \Phi_1 L \dots - \Phi_p L^p$ . The joint distribution of the process depends on parameters  $C, \Phi_1, \dots, \Phi_p$  through the first and second-order moments of the process, or equivalently through the matrix spectral density :

$$f(w) = \frac{1}{2\pi} \Phi(\exp iw) C C' \Phi(\exp(-iw))'. \quad (2.19)$$

Several pairs  $\Phi(L), C$  yield the same spectral density, and, therefore, observationally equivalent Gaussian processes, leading to two different identification issues :

i)  $CC'$  can be identified, but not  $C$  itself.

ii) The different polynomials in  $\Phi(L)$  are deduced from the fundamental solution (i.e. such that the roots of  $\det \Phi(z)$  are outside the unit circle) by an appropriate use of Blaschke matrices [see e.g., Hansen, Sargent (1981), Lippi, Reichlin (1994)].

Box-Jenkins approach consists in estimating parameters  $\Omega = CC', \Phi_1, \dots, \Phi_p$  by maximizing the Gaussian log-likelihood (if  $\eta_t$  is Gaussian), or the pseudo log-likelihood (if  $\eta_t$  is non-Gaussian) :

$$(\hat{\Phi}, \hat{\Omega}) = \arg \max_{\Phi, \Omega} \sum_{t=1}^T \left\{ -\frac{n}{2} \log 2\pi - \frac{1}{2} \log \det \Omega - \frac{[\Phi(L)Y_t]' \Omega^{-1} [\Phi(L)Y_t]}{2} \right\}. \quad (2.20)$$

In other words  $\hat{\Phi}$  is obtained from the O.L.S. estimators equation by equation, and  $\hat{\Omega}$  is the empirical variance-covariance matrix of the residuals.

These (pseudo) maximum likelihood estimators converge when  $T$  tends to infinity, and the estimated autoregressive polynomial  $\hat{\Phi}(L)$  converges to the fundamental solution <sup>14</sup> associated with  $\Phi(L)$  (i.e. by inverting in an appropriate way the ill-located roots through Blaschke matrices and orthonormal transformations <sup>15</sup>). Therefore, it is not consistent whenever the true VAR

<sup>14</sup>This result is well-known, if the errors have finite variance. It is also valid for errors with fat tails [see Davis, Resnick (1986)].

<sup>15</sup>A Blaschke matrix is a square matrix of the lag operator  $B(L)$  such that  $[B(L)]^{-1} = B^*(L^{-1})$  where  $B^*(.)$  is obtained from  $B(.)$  by transposing and taking conjugate coefficients.



defined in (2.18) features nonfundamentality.

To adjust for this lack of consistency, it is proposed in the literature to deduce all the remaining solutions by applying the Blaschke matrices <sup>16</sup> [see Lippi, Reichlin (1991)].

### 3 The identification issues in the non Gaussian SVARMA

Let us consider the VARMA model,

$$\Phi(L)Y_t = \Theta(L)C\eta_t, \quad (3.1)$$

$$\text{where : } \Phi(L) = Id - \Phi_1L - \dots - \Phi_pL^p, \Theta(L) = Id - \Theta_1L - \dots - \Theta_qL^q, \quad (3.2)$$

and the components  $\eta_{1t}, \dots, \eta_{nt}$  of the error term are independent (and also serially independent by Assumption a.2).

This additional cross-sectional independence assumption is needed for impulse response analysis. Let us for instance assume that  $\eta_{1t}$  has an economic interpretation, such as a technological shock. To ensure that a change in  $\eta_{1t}$  captures only the effect of this shock, it is necessary to eliminate any link between  $\eta_{1t}$  and the other components, i.e. to assume their independence.

In this respect we follow the structural VARMA literature by assuming that all structural shocks are serially and mutually orthogonal. But we emphasize that the appropriate notion of "orthogonality" is independence. Indeed the absence of correlation usually considered in the SVARMA literature is not sufficient for deriving the impulse response function and its confidence intervals. It is also insufficient for comparing the impulse response functions deduced from a nonlinear DSGE and from its SVARMA approximation, or for applying Bayesian techniques à la Sims-Litterman. Distributional assumptions are required. <sup>17</sup> We point out below that both identification challenges

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<sup>16</sup>See Leeper et al. (2013), p1123-1124 for a practical example of use of Blaschke matrices.

<sup>17</sup>For all these problems, the shocks are usually implicitly assumed Gaussian, but the Gaussian hypothesis is never tested in practice. This assumption is even explicit in some papers [see e.g. Forni et alii (2013)].

are solved when the errors are independent, with at most one Gaussian error. We discuss first the static case, then the dynamic case.

### 3.1 How to orthogonalize the shocks

Let us first consider a static framework. When the noise is Gaussian  $\eta_t \sim N(0, Id)$ , we cannot identify  $C$  and  $\eta_t$  given the knowledge of  $C\eta_t$ . Indeed we have :

$$C\eta_t = C^*\eta_t^*, \text{ with } \eta_t^* \sim N(0, Id),$$

as soon as :

$$C^* = CQ \text{ and } \eta_t^* = Q'\eta_t,$$

where  $Q$  is an orthogonal matrix :  $QQ' = Id$ .

Thus, in a Gaussian framework, there are several ways of selecting the  $\eta_t$ 's and thus of "orthogonalizing" the shocks. These different possibilities underlie the recursive identification scheme proposed by Sims (1980), (1989).

<sup>18</sup> Additional structural short run restrictions [see e.g. Bernanke (1980), Sims (1986), Rubio-Ramirez, Waggoner, Zha (2010)], and long run restrictions <sup>19</sup> are sometimes introduced in the applied literature to reduce the set of possibilities [see e.g. Blanchard, Quah (1989), Faust, Leeper (1997), Erceg, Guerrieri, Gust (2005), Christiano, Eichenbaum, Vigfusson (2007)], as well as sign restrictions [Uhlig (2005), Chari, Kehoe, McGrattan (2008), Mountford, Uhlig (2009)].<sup>20</sup>

On the contrary, there is no identification problem in a non Gaussian framework as shown in the following Proposition [see Comon (1994), Theorem 11 for square integrable variable, Eriksson, Koivunen (2004), Theorem 3 ii) and the references therein, for variables with fat tails] :

**Proposition 1 :** Let us consider two vectors  $\eta_t, \eta_t^*$  of the same size  $n$ , with independent components, with continuous distributions and satisfying a

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<sup>18</sup>See also Klein (2000).

<sup>19</sup>Typically permanent shocks to output are associated with technology shocks.

<sup>20</sup>An alternative consists in leaving the linear dynamic framework by considering Markov Switching SVAR [see Lanne et al. (2010), Lutkepohl (2013), Herwatz, Lutkepohl (2014), Velinov, Chen (2013)]. This extended framework allows to test the identification restrictions. In this note we will stay in a pure SVARMA framework.

linear relationship :  $\eta_t^* = M\eta_t$ , where  $M$  is invertible. Then, these component are such that  $\eta_{i,t}^* = \sigma_i \eta_{\pi(i),t}$ ,  $i = 1, \dots, n$ , where  $\pi$  is a permutation of  $\{1, \dots, n\}$ , whenever at most one component is Gaussian.

The proof is a direct consequence of the Darmois, Skitovich characterization of the multivariate normal distribution [see Darmois (1953), Skitovich (1953), Ghurye, Olkin (1961), Theorem p 533, Kagan, Linnik, Rao (1973), Th 10.3.1]. This identification result underlies independent component analysis (ICA), which is the analogue of the principal component analysis (PCA) when components are required to be independent rather than being simply uncorrelated [see e.g. Hyvarinen, Karhunen, Oja (2001)]. In practice consistent estimators of matrix  $C$  are obtained by considering cross-moment conditions, or tail properties.

An important consequence of Proposition 1 is the following : Let us consider the error terms  $\varepsilon_t = C\eta_t$  in the VARMA model (3.1), with independent components for the  $\eta_t$ , at most one Gaussian component and a lower triangular matrix  $T$  (not diagonal). Then the components of  $T\eta_t$  cannot be mutually independent except in a very special case. Indeed the random vectors  $\eta_t^* = T\eta_t$  and  $\eta_t$  satisfy a one-to-one linear relationship  $\eta_t^* = TC\eta_t$ . Thus by Proposition 1, we deduce that :  $TC = \Lambda P$ , where  $\Lambda$  is a diagonal matrix and  $P$  a permutation matrix. Thus matrix  $C$  must be the product of a triangular matrix by a permutation matrix, which is a very specific situation. **This implies that, in a non-Gaussian framework, the recursive scheme proposed by Sims can be used to find uncorrelated components, but not, in general, independent components.**

### 3.2 Identification of the relevant nonfundamental representation

We have seen in Section 3.1 above that, in a non-Gaussian framework, there is no real identification problem for the "static" part of the SVARMA model. What can be said about the identification of its dynamic part? The question concerns the uniqueness of the two-sided moving average representation :

$$Y_t = \sum_{j=-\infty}^{+\infty} A_j \eta_{t-j}, \text{ with } \sum_{j=-\infty}^{+\infty} \|A_j\| < \infty, \quad (3.3)$$

where the components of  $\varepsilon_t$  are mutually independent. The following result (Chan, Ho, Tong (2006), Theorem 1) extends the Darmais, Skitovich result to Gaussian processes <sup>21</sup> and is valid under the assumption of nondegenerate transfer function :

**Assumption a.6 :** The determinant of the transfer function

$\tilde{A}(w) = \sum_{j=-\infty}^{+\infty} A(j) \exp(-ijw)$  is not zero almost everywhere on the interval  $(-\pi, \pi)$ .

This assumption is introduced to avoid linear dependence between the columns of matrix polynomial  $A(L)$ . If, for instance  $y_t = A(L)\eta_{1t} + A_2(L)\eta_{2t}$ , say, with  $A_2(L) = \lambda A_1(L)$ , we can write  $y_t = A_1(L)(\eta_{1t} + \lambda\eta_{2t})$ , and replace the bidimensional noise  $(\eta_{1t}, \eta_{2t})$  by the one dimensional noise  $\eta_{1t} + \lambda\eta_{2t}$ .

**Proposition 2 :** Under Assumption a.6 of nongedenerate transfer function, let us consider two moving average representations of a non Gaussian process  $(Y_t)$  :

$$Y_t = \sum_{j=-\infty}^{+\infty} A_j \eta_{t-j} = \sum_{j=-\infty}^{+\infty} A_j^* \eta_{t-j}^*, \forall t.$$

Then

$$\eta_{i,t-m(i)}^* = \sigma_i \eta_{\pi(i),t}, A_{\pi(i),j-m(i)} = A_{i,j}^* \sigma_i,$$

where  $\eta_{i,t}$  is the  $i^{th}$ -component of  $\eta_t$  and  $A_{i,j}$  the  $i^{th}$ -column of  $A_j$ , for some scalars  $\sigma_i$ , integers  $m(i)$  and permutation  $\pi$  of the set  $\{1, 2, \dots, m\}$ , if one of the two following conditions is satisfied :

Condition  $C_1$  : the components of  $\eta_t$  are identically distributed.

Condition  $C_2$  : each component of  $\eta_t$  has a nonzero  $r^{th}$  cumulant, with  $r \geq 3$ , and a finite moment of order  $s$  where  $s$  is an even integer greater than <sup>22</sup>  $r$ .

Thus, the two-sided moving average representation is unique up to a permutation, a change of scale and a time shift, possibly depending on the component.

<sup>21</sup>See Findley (1986), Cheng (1992) for the one-dimensional case  $n = m = 1$ .

<sup>22</sup>Note that condition C2 implies that all the components of  $\eta_t$  are not Gaussian.

To understand how Proposition 2 solves the identification issue, let us consider a bivariate MA(1) model :

$$\begin{cases} y_{1t} &= \eta_{1t} - \theta_{1,1}\eta_{1,t-1} - \theta_{1,2}\eta_{2,t-1}, \\ y_{2t} &= \eta_{2t} - \theta_{2,1}\eta_{1,t-1} - \theta_{2,2}\eta_{2,t-1}. \end{cases}$$

where  $\eta_{1,t}, \eta_{2,t}, t$  varying, are identically distributed and non-Gaussian. Then the other moving average representations with i.i.d. components of the error terms are either of the type :

$$\begin{cases} y_{1t} &= \eta_{1,t-m(1)} - \theta_{1,1}\eta_{1,t-1-m(1)} - \theta_{1,2}\eta_{2,t-1-m(2)}, \\ y_{2t} &= \eta_{2,t-m(2)} - \theta_{2,1}\eta_{1,t-1-m(1)} - \theta_{2,2}\eta_{2,t-1-m(2)}, \end{cases}$$

or of the type :

$$\begin{cases} y_{1t} &= \eta_{2,t-m(2)} - \theta_{1,1}\eta_{2,t-1-m(2)} - \theta_{1,2}\eta_{1,t-1-m(1)}, \\ y_{2t} &= \eta_{1,t-m(1)} - \theta_{2,1}\eta_{2,t-1-m(2)} - \theta_{1,2}\eta_{2,t-1-m(2)}. \end{cases}$$

Thus they differ only by a redenomination of the errors.

A similar identification result has been recently derived when the components of  $\eta_t$  have fat tails [see Gouriéroux, Zakoian (2014)b], and is applicable for rational expectation models with non square integrable solutions [see Section 2.3 iii)].<sup>23</sup>

Proposition 2 has far reaching consequences. In particular **if a non-Gaussian stationary process has a VARMA representation with serially independent errors, this representation is unique and all the second order equivalent representations have serially uncorrelated, but not independent, errors.**

Proposition 2 also implies that there exist better estimation methods than the Gaussian pseudo maximum likelihood used in the BJ methodology

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<sup>23</sup>Note that the identification result in Chen, Choi, Escanciano (2012), Theorem 1, is much less powerful. This result provides condition to check if the fundamental representation is the right one, but cannot be used to find the correct nonfundamental representation, otherwise.

(see Section 4). These alternative methods will provide consistent estimators of the true two-sided moving average polynomial  $A(L)$  [or equivalently the true nonfundamental  $\Phi(L), \Theta(L)$  in the VARMA representation]. They also provide consistent nonparametric estimators of the distribution of the components of the error term.

When the distribution of the error term is parametrically specified, maximum likelihood approaches<sup>24</sup> and appropriate Bayesian approaches can be used, even with ill-located roots [see e.g. Andrews, Breidt, Davis (2006), Lanne, Saikkonen (2013), Davis, Song (2012), Gouriou, Jasiak (2014) for ML, Lanne, Luoma, Luoto (2012) for Bayesian approach] [and Appendix 2 for the main steps of the ML and Bayesian procedures]. However, these approaches require to compute the likelihood functions in each regime of ill-located roots.<sup>25</sup>

## 4 No need of Blaschke matrices for statistical inference

We have noted in Section 3 and Appendix 2 the difficulty in implementing maximum likelihood or Bayesian approaches, for possible nonfundamental SVARMA models. Indeed the (approximated) log-likelihood has different expressions according to the location of the roots of the autoregressive or moving average polynomials (see also the discussion in Appendix 3 i)).

In this section we introduce new parametric and semi-parametric consistent estimation methods which circumvent the use of Blaschke matrices. We try to get simple estimation methods by avoiding nonlinear optimizations with respect to a large number of parameters. This explains why we

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<sup>24</sup>Called approximate maximum likelihood (AML) in the literature on noncausal time series.

<sup>25</sup>The need for distinguishing the different regimes is not specific to linear dynamic models, but also arise in nonlinear autoregressive models, say, whenever the process of interest is assumed strictly stationary. As an illustration let us consider a one dimensional autoregressive model :  $y_t = a(y_{t-1}, \varepsilon_t)$ , say, where  $a$  is invertible with respect to  $y_{t-1}$ . So we can also write  $y_{t-1} = b(y_t, \varepsilon_t)$ , with  $b = a^{-1}$ . By recursive substitution we can write  $y_t$  function of  $\varepsilon_t, \varepsilon_{t-1}, \varepsilon_t, y_{t-k-1}$ , or  $y_t$  function of  $\varepsilon_{t+1}, \dots, \varepsilon_{t+h}, y_{t+h}$ . By considering a large value of  $h$ , only one of the two nonlinear moving-average representations of  $y_t$  will exist. Thus the backward or forward expressions of the stationary solution ( $Y_t$ ) depends on the stability properties of function  $a$  and  $b = a^{-1}$ .

focus on multistep approaches and use when possible instrumental variable approaches. The cost of the simplicity is a reasonable loss of efficiency.

We first consider the parametric framework of a SVARMA model with nonfundamentalness in the moving average dynamics only and introduce a two-step Incomplete Maximum Likelihood (IML) approach. Then we explain how to estimate semi-parametrically a SVAR nonfundamental representation by an appropriate instrumental variable (IV) approach. Since in macroeconomic applications the number of observations is not very large, it is preferable to measure the accuracy of the estimators by bootstrap, not by using the theoretical formulas of asymptotic variance-covariance matrices derived from the econometric literature. For this purpose we explain how to simulate the path of a mixed causal/noncausal autoregressive process.

## 4.1 How to simplify parametric inference for nonfundamentalness in the moving average dynamics.

The likelihood function of a SVARMA with possible ill-located roots has a complicated expression with several regimes of roots to be taken into account via Blaschke transformations (see Appendix 2). In fact parametric estimation can be greatly simplified if nonfundamentalness concerns the moving-average dynamics only, as it is often the case in structural macromodels. To introduce the estimation approach, we first discuss the case of a one-dimensional MA(1) process before considering the general framework of a VARMA process.

### 4.1.1 The one-dimensional MA(1) process

Let us consider a one-dimensional MA(1) process :

$$y_t = \varepsilon_t - \theta\varepsilon_{t-1}, \quad (4.1)$$

where the  $\varepsilon_t$ 's are independent.

Suppose that we observe  $y_1, \dots, y_T$ . If the common distribution of the  $\varepsilon_t$ 's is  $N(0, \sigma^2)$ , the model is not identifiable. Indeed the distribution of  $(y_1, \dots, y_T)$  is multivariate normal, with zero mean and a variance-covariance matrix defined by  $V(y_t) = \sigma^2(1+\theta^2)$ ,  $cov(y_t, y_{t-1}) = -\sigma^2\theta$ , and  $cov(y_t, y_{t-h}) = 0, \forall h \geq 2$ . Obviously the two sets of parameters  $(\theta, \sigma^2)$  and  $(\frac{1}{\theta}, \sigma^2\theta^2)$  give the same distribution.

If  $\varepsilon_t$  is not Gaussian, Proposition 2 shows that the model is identifiable<sup>26</sup>. Let us denote by  $g(\varepsilon; \gamma)$  the common p.d.f. of the  $\varepsilon'_t$ s, where  $\gamma$  is an unknown parameter.

i) When  $|\theta| < 1$ , we can invert equation (4.1) in the standard way in order to get  $\varepsilon_t$  as a function of current and lagged values of process  $Y$  as :

$$\varepsilon_t = \sum_{h=0}^{\infty} \theta^h y_{t-h}. \quad (4.2)$$

Then the log-likelihood function is approximated by :

$$L_1^a(\theta, \gamma) = \sum_{t=1}^T \log g\left(\sum_{h=0}^{t-1} \theta^h y_{t-h}; \gamma\right), \quad (4.3)$$

where the infinite sums are truncated to be compatible with the observed  $y_1, \dots, y_T$ .

ii) When  $|\theta| > 1$ , equation (4.1) can still be inverted, but in reversed time. We get :

$$\begin{aligned} y_t &= \varepsilon_t - \theta \varepsilon_{t-1} \\ \Leftrightarrow -\frac{y_{t+1}}{\theta} &= \varepsilon_t - \frac{1}{\theta} \varepsilon_{t+1} \\ \Leftrightarrow \varepsilon_t &= -\sum_{h=0}^{\infty} \frac{1}{\theta^{h+1}} y_{t+h+1}. \end{aligned} \quad (4.4)$$

The log-likelihood function is approximated by :

$$L_2^a(\theta, \gamma) = \sum_{t=1}^T \log \left\{ \frac{1}{|\theta|} g\left(-\sum_{h=0}^{T-t-1} \frac{1}{\theta^{h+1}} y_{t+h+1}; \gamma\right) \right\}, \quad (4.5)$$

where the sums are now truncated to account for the most recent observation and factor  $1/|\theta|$  comes from the Jacobian formula.

iii) Simple formulas do not exist when  $|\theta| = 1$ .

Then the practice of ARMA modelling consists in assuming the invertibility, i.e.  $|\theta| \neq 1$ , and in considering the approximated log-likelihood function :

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<sup>26</sup>See Appendix 3 for a more detailed discussion of non-identifiability of a MA(1) process and the links with reversibility.



$$L^a(\theta, \gamma) = L_1^a(\theta, \gamma)\mathbb{1}_{|\theta|<1} + L_2^a(\theta, \gamma)\mathbb{1}_{|\theta|>1}. \quad (4.6)$$

The optimization is performed in two steps :

First step : we optimize separately the log-likelihood on each regime to get :

$$(\hat{\theta}_1, \hat{\gamma}_1) = \arg \max_{\theta, \gamma, |\theta|<1} L_1^a(\theta, \gamma),$$

$$(\hat{\theta}_2, \hat{\gamma}_2) = \arg \max_{\theta, \gamma, |\theta|>1} L_2^a(\theta, \gamma).$$

Second step : the approximated ML estimator is :

$$\begin{aligned} (\hat{\theta}, \hat{\gamma}) &= (\hat{\theta}_1, \hat{\gamma}_1), \text{ if } \hat{L}_1^a = L_1^a(\hat{\theta}_1, \hat{\gamma}_1) > \hat{L}_2^a = L_2^a(\hat{\theta}_2, \hat{\gamma}_2), \\ &= (\hat{\theta}_2, \hat{\gamma}_2), \text{ otherwise.} \end{aligned}$$

The approach above has at least three drawbacks.

i) First, we do not know how to simply approximate the log-likelihood function in the noninvertible case  $|\theta| = 1$ .

ii) Second, this approach is difficult to implement in the multidimensional case, when the number of regimes for ill-located roots increases and the expressions of  $\varepsilon_t$  as two sided autoregressive functions of  $y_t$  have to be derived. (see Appendix 2).

iii) Third, the focus on the regimes when approximating the log-likelihood function gives the misleading impression of a lack of continuity of the exact log-likelihood function w.r.t.  $\theta$  at  $|\theta| = 1$ , whereas this exact log-likelihood is continuous.<sup>27</sup> Let us derive it for the discussion. We get :

$$\varepsilon_1 = y_1 + \theta\varepsilon_0, \varepsilon_2 = y_2 + \theta y_1 + \theta^2\varepsilon_0, \dots, \varepsilon_T = y_T + \theta y_{T-1} + \dots + \theta^{T-1}y_1 + \theta^T\varepsilon_0.$$

Thus the joint p.d.f. of  $y_1, \dots, y_T$  given  $\varepsilon_0$  is :

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<sup>27</sup>Such an exact log-likelihood is for instance used in the Gaussian case, with  $|\theta| < 1$  by Chen, Davis, Song (2011) to analyze the properties of the ML estimator of a moving-average parameter close to noninvertibility.

$$\prod_{t=1}^T g\left(\sum_{h=0}^{t-1} \theta^h y_{t-h} + \theta^t \varepsilon_0; \gamma\right),$$

and the exact log-likelihood is :

$$L(\theta, \gamma) = \log\left\{ \int \prod_{t=1}^T g\left(\sum_{h=0}^{t-1} \theta^h y_{t-h} + \theta^t \varepsilon; \gamma\right) g(\varepsilon; \gamma) d\varepsilon. \right\} \quad (4.7)$$

The exact log-likelihood is generally a differentiable function of  $\theta$ . However, its expression is not appropriate for deriving the asymptotic properties of the ML estimators. For such derivation, it is usually shown that it can be approximated by appropriate sums, that are  $L_1^a(\theta, \gamma)$ , or  $L_2^a(\theta, \gamma)$  in our example, sums for which the standard asymptotic theories apply [see e.g. Lanne, Saikkonen (2013) for this approach applied to VAR models with non-fundamentalness].

The approaches proposed in the next subsection are based on optimization criteria, which can be decomposed into sums appropriate in order to apply standard asymptotic theory and are sufficiently regular to avoid the introduction of the different regimes of ill-located roots.

#### 4.1.2 Incomplete Maximum Likelihood (IML), PML and SPML approaches

The principle of Incomplete Maximum Likelihood (IML) is easily explained in the case of the MA(1) process discussed above.

Let us separate the observations by throwing away an observation every third observation. The set of observations becomes :

$$y_1, y_2, y_4, y_5, \dots, y_{3j-1}, y_{3j-2}, \dots$$

We have a loss of information since observations  $y_3, y_6, \dots, y_{3j}, \dots$  are not taken into account with the advantage that the pairs of observations  $(y_{3j-1}, y_{3j-2}), j$  varying, are i.i.d. Thus the exact log-likelihood function corresponding to these incomplete observations is easily computed, naturally decomposed as a sum and the standard asymptotic theory applies.

Let us now describe how the IML approach can be adapted to the multidimensional framework. For expository purpose, we consider a VARMA (1,1) model :

$$Y_t - \Phi Y_{t-1} = \varepsilon_t - \Theta \varepsilon_{t-1}, \quad (4.8)$$

where the errors  $\varepsilon_t$  are i.i.d., not Gaussian, with p.d.f.  $g(\varepsilon, \Gamma)$ . The joint distribution of the errors is chosen such that :

$$\varepsilon_t = C\eta_t, \text{ say,}$$

where the  $\eta_t$  are independent, with a same distribution with parameter  $\gamma$ , say; thus  $\Gamma = (C, \gamma)$ . Such a choice of the joint distribution of the components of  $\varepsilon_t$  is required in order to get mutually independent shocks when defining the impulse response functions.<sup>28</sup>

The roots of  $\det \Phi(z)$  are assumed to be outside the unit circle, but the roots of  $\det \Theta(z)$  can be anywhere, inside, outside, or even on the unit circle.

Model (4.8) is a Seemingly Unrelated Regression (SUR) model :

$$Y_t = \Phi Y_{t-1} + v_t, \quad (4.9)$$

and the autoregressive matrix  $\Phi$  can be estimated by instrumental variable (IV), using as instruments  $Y_{t-2}$  ( $Y_{t-3}, Y_{t-4}, \dots$ ), which are uncorrelated with  $v_t$ . Let us denote  $\hat{\Phi}$  the corresponding IV estimator.

Next, let us consider the joint p.d.f. of  $v_t = \varepsilon_t - \Theta \varepsilon_{t-1}$ ,  $v_{t-1} = \varepsilon_{t-1} - \Theta \varepsilon_{t-2}$ . This p.d.f. is given by :

$$h(v_t, v_{t-1}; \Theta, \Gamma) = \int g(v_t + \Theta v_{t-1} + \Theta^2 \varepsilon; \Gamma) g(v_{t-1} + \Theta \varepsilon; \Gamma) g(\varepsilon; \Gamma) d\varepsilon. \quad (4.10)$$

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<sup>28</sup>It is often assumed that the distribution of the error term  $\varepsilon_t$  of the VAR belongs to the standard multivariate Student family, but this assumption is not appropriate for the analysis of impulse responses since this family does not include the case of independent components. We can assume, for instance, that the  $i^{th}$  component of  $\eta_t$ , follows a univariate Student distribution with  $\nu(i)$  degrees of freedom. Additional identification restrictions can be introduced to fix the denomination of the errors, i.e. to solve the problem of multiplicity by change of scale and permutations. For instance, the change by permutation is excluded, if we impose either that the degrees of freedom  $\nu(i)$  are in an increasing order, or that the elements of the first row of matrix  $C$  are in an increasing order. The second condition seems preferable, since it is compatible with the limiting case of equal degrees of freedom.

The two step IML estimator of  $(\Theta, \Gamma)$  is the solution of :

$$(\hat{\Theta}, \hat{\Gamma}) = \arg \max_{\Theta, \Gamma} \sum_{j=1}^{J=(T/3)} \log h(Y_{3j-1} - \hat{\Phi}Y_{3j-2} - \hat{\Phi}Y_{3j-3}; \Theta, \Gamma).$$

The two step IML estimator has standard asymptotic properties, irrespective of the location of the roots of  $\det \Theta(z)$ . It is in particular consistent, asymptotically normal. and its asymptotic variance-covariance matrix can be derived.

All the observations of  $(y_t)$  are used if  $\Phi \neq 0$ , but in a non optimal way. Other consistent estimators as simple to implement and using observations in a more efficient way can be based on the same idea.

We can consider the estimator solution of :

$$(\tilde{\Theta}, \tilde{\Gamma}) = \arg \max_{\Theta, \Gamma} \sum_{t=2}^T \log h(Y_t - \hat{\Phi}Y_{t-1}, Y_{t-1} - \hat{\Phi}Y_{t-2}; \Theta, \Gamma). \quad (4.11)$$

This two step Pseudo Maximum Likelihood (PML) estimator is using the information on all the  $v_t$ 's [see e.g. Gouriou, Monfort, Trognon (1985)]. It is also consistent, asymptotically normal, but the asymptotic variance-covariance matrix is now computed by a sandwich formula (also involving a general central limit theorem).

The IML and PML likelihood functions depend on integrals of the same dimension as the VAR system. We can approximate the integral in function  $h$  by simulation to get a two-step Simulated Pseudo Maximum Likelihood (SPML) estimator. The numerical optimization of the approximated criterion is :

$$(\Theta^*, \Gamma^*) = \arg \max_{\Theta, \Gamma} \sum_{t=1}^T \log \hat{h}(Y_t - \hat{\Phi}Y_{t-1}, Y_{t-1} - \hat{\Phi}Y_{t-2}; \Theta, \Gamma), \quad (4.12)$$

where :

$$\hat{h}(v_t, v_{t-1}; \Theta, \Gamma) = \frac{1}{S} \sum_{s=1}^S \{g(v_t + \Theta v_{t-1} + \Theta^2 \varepsilon^s(\Gamma), g(v_{t-1} + \Theta \varepsilon^s(\Gamma); \Gamma)\}, \quad (4.13)$$

and the  $\varepsilon^s(\Gamma)$  are drawn independently in distribution  $g(\varepsilon; \Gamma)$ .<sup>29</sup>

When the number  $S$  of simulations tends to infinity sufficiently fast w.r.t. the number  $T$  of observations, the SPML estimator has the same asymptotic properties as the PML estimator itself [see Gouriéroux, Monfort (1996) for a general presentation of Simulation Based Estimation Methods].

**Remark 1 :** The objective functions used in the IML or PML approaches are well-specified and misspecified log-likelihood functions, respectively. Thus they can be used to develop Bayesian approaches [see Muller (2013) for Bayesian approaches with misspecified models].

**Remark 2 :** The IML , PML, SPML approaches provide subefficient consistent estimators. Even if the lack of efficiency is expected reasonable, we can look for efficient estimators. This is done as follows:

First step : Apply a IML (PML, SPML) approach. This provides a consistent estimator  $\hat{\Theta}$  of  $\Theta$  and of the true regime of nonfundamentalness. Then we can derive the expression of the (approximated) log-likelihood function  $L_s^a(\Phi, \Theta, \Gamma)$  corresponding to this regime  $s$ . In a second step, we maximize this approximated  $L_s^a$  to derive the efficient ML estimator. The advantage of this two-step approach compared to the first step ML approach is to focus on one regime instead of a large number. This two step approach is a kind of pre-test estimation approach, in which the one step IML (PML, SPML) is used to detect the right regime of roots.

**Remark 3 :** Why not consider a simulated maximum likelihood (SML) approach ? The SML estimators would be defined as :

$$(\hat{\Theta}, \hat{\Gamma}) = \arg \max_{\Theta, \Gamma} \log \left\{ \frac{1}{S} \sum_{s=1}^S \prod_{t=1}^T g \left( \sum_{h=0}^{t-1} \Theta^h (Y_{t-h} - \hat{\Phi} Y_{t-h-1}) + \Theta^t \varepsilon^s(\Gamma); \Gamma \right) \right\}.$$

However, the expression of the objective function has been derived backward, which implies terms like  $\Theta^t \varepsilon^s(\Gamma)$ . If the true representation is nonfundamental and  $T$  is rather large,  $\Theta^t$  will have exploding components when  $t$  is increasing. Thus the SML estimator will be very sensitive to drawings of simulated  $\varepsilon$  in the tail and not robust. The IML approach and its extensions do not have this drawback.

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<sup>29</sup>As usual the same basic drawings must be kept when  $\Gamma$  is modified in the optimization algorithm.

## 4.2 Semi-parametric estimation of a SVAR model with nonfundamental representation.

In this section we consider semi parametric estimation methods, i.e. methods which do not make parametric assumptions on the distribution of the errors.

For expository purpose, let us consider a SVAR(1) model :

$$Y_t = \Phi Y_{t-1} + \varepsilon_t, \quad (4.14)$$

with i.i.d. non Gaussian error terms, with zero mean.<sup>30</sup>

The BJ approach, based on a Gaussian pseudo-likelihood, estimates the Seemingly Unrelated Regression model (4.14) by ordinary least squares, or equivalently by an instrumental variable approach with instrument  $Y_{t-1}$ . This method is valid if  $Y_{t-1}$  and  $\varepsilon_t$  are uncorrelated, that is, if all the roots of  $\det \Phi(z)$  are outside the unit circle. It is not valid if some roots are inside the unit circle, since  $Y_t$  has a two-sided moving average representation that creates correlation between  $Y_{t-1}$  and  $\varepsilon_t$ .

However the serial independence between the errors implies moment restrictions of the type :

$$Cov[a(Y_t - \Phi Y_{t-1}), b(Y_{t-h} - \Phi Y_{t-h-1})] = 0, \quad (4.15)$$

for any  $h$ , and any square integrable functions  $a$  and  $b$ . These restrictions can be the basis for covariance estimators. In particular, for a square integrable process  $Y_{t-1}$ <sup>31</sup> we have :

$$E[(Y_t - \Phi Y_{t-1})b(Y_{t-1} - \Phi Y_{t-2})] = 0, \text{ for any square integrable function } b. \quad (4.16)$$

In other words, the variables  $b(Y_{t-1} - \Phi_0 Y_{t-2})$ , where  $\Phi_0$  is the true value of the autoregressive matrix, are valid instruments for the SVAR model (4.14), and the moment conditions (4.16) are used to estimate jointly  $\Phi$  and these instruments.

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<sup>30</sup>Other moment restrictions can be used for MA processes. See the discussion in Appendix 3 iii).

<sup>31</sup>When the process has fat tails as in the case of explosive bubble solutions of rational expectation models in Section 2.2, we have to select  $a$  and  $b$  functions, such as  $a(y) = \exp(ia'y)$ ,  $b(y) = \exp(ib'y)$ , say.

In practice a set of functions  $b$  has to be selected to apply the Generalized Method of Moments (GMM). When choosing only the identity function, the identification issue is likely not solved.<sup>32</sup> Other functions, such as polynomial functions of degree 2, 3, have to be added to identify which nonfundamental representation is the right one,

The GMM can be applied in several steps.

i) Step 1 : Consistent estimation.

First calibrate parameter  $\Phi$  from a set of moment conditions without trying to introduce optimal weights. The corresponding estimator  $\tilde{\Phi}$  is consistent and can be used to deduce approximated instrumental variables  $\tilde{b}(Y_{t-1} - \tilde{\Phi}Y_{t-2})$ , where the set of  $\tilde{b}$  functions can be larger than  $b$ .

ii) Step 2 : Efficiency improvement.

Apply to the model :

$$Y_t = \Phi Y_{t-1} + \varepsilon_t,$$

a Two Stage Instrumental Variable (2SIV) approach based on instruments  $\tilde{b}(Y_{t-1} - \tilde{\Phi}Y_{t-2})$ . This second step provides closed form estimators once  $\tilde{\Phi}$  has been derived and avoids an additional nonlinear optimization.

iii) Step 3 : Diagnostic tools for serial independence.

Let us denote  $\hat{\varepsilon}_t = Y_t - \hat{\Phi}Y_{t-1}$ , the associated residuals, where  $\hat{\Phi}$  denotes the 2SIV estimator. We have to check if the vectors  $\varepsilon_t$  are serially independent. We can perform tests of serial independence, based for instance on power cross-moments at any lag, for all these estimated shocks. The standard analysis of ACF and cross ACF of BJ has to be completed by considering ACF and cross ACF on nonlinear transforms of the  $\hat{\varepsilon}_t$ . Typically square ACF and square cross ACF have to be considered to detect possible ARCH effects creating nonlinear dependence, or cross ACF between  $(\hat{\varepsilon}_t$  and  $(\hat{\varepsilon}_{t-h}^2)^2$  to detect leverage effects, and so on.<sup>33</sup>

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<sup>32</sup>In the univariate case, it is easily seen that  $\varphi$  satisfies the second order equation  $\varphi_0 - \varphi(1 + \varphi_0^2) + \varphi^2\varphi_0 = 0$ , where  $\varphi_0$  is the true value of  $\varphi$  and this equation has two solutions  $\varphi_0$  and  $1/\varphi_0$ . Moreover, adding the condition  $E[(y_t - \varphi y_{t-1})(y_{t-1} - \varphi y_{t-2})^2] = 0$ , we get a third order equation in  $\varphi$  which admits  $\varphi_0$  as a solution, but not  $1/\varphi_0$ , in general.

<sup>33</sup>The generalized spectrum approach proposed in Hong (1999), Chen et al. (2012) considers complex exponential transforms  $\exp(ia'\hat{\varepsilon}_t)$  and  $\exp(ib'\hat{\varepsilon}_{t-h})$ , say.

If the serial independence hypothesis is rejected, the SVAR model cannot be used for computing impulse response functions. Otherwise, we have still to check if the vector of errors depend linearly of independent "structural" shocks.

iv) Step 3 : Analysis of mutual independence.

The hypothesis of mutual independence can be written as :

$$H_0 = \{ \exists C: \varepsilon_t = C\eta_t, \text{ the components of } \eta_t \text{ being independent} \}.$$

The test of this hypothesis and the estimation of the  $C$  matrix can be done by applying Independent Component Analysis [see e.g. Hyvarinen, Karhunen, Oja (2001), Hyvarinen et al. (2008), Moneta et al. (2013)].

If this hypothesis is rejected, the SVAR model is misspecified. Otherwise we know how to estimate a  $C$  matrix by ICA, that is how to "orthogonalize" the components of  $\varepsilon_t$ .<sup>34</sup>

### 4.3 Simulation

Even it is possible to derive the asymptotic variance-covariance estimators of the different estimators introduced in Sections 4.1, 4.2, the rather limited number of observations in macroeconomic applications make preferable to measure the accuracy by bootstrap. To apply the bootstrap, we have to know how to simulate the path of a SVARMA with nonfundamental representation in either the autoregressive or the moving average components.

i) The SVARMA (1,1) parametric model

Let us first consider the model of Section 4.1 and denote  $\hat{\Phi}, \hat{\Theta}, \hat{\Gamma}$  the estimated parameters. In this case the simulation is straightforward:

- First draw independently  $\varepsilon_t^s$  in the distribution  $g(\varepsilon, \hat{\Gamma})$ ;
- Then compute the simulated  $w_t^s = \varepsilon_t^s - \hat{\Theta}\varepsilon_{t-1}^s$

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<sup>34</sup>This multistep IV approach can be compared with extension of the LiNGAM [Linear Non-Gaussian Acyclic Model] maximum likelihood approach introduced in Hyvarinen et al. (2010). The authors propose to maximize a pseudo log-likelihood, where the p.d.f. of the independent shock components of  $\eta$  are of the type :  $\log g(\eta) = -\sqrt{2}|\eta| + \text{const.}$  [see eq.9, in Hyvarinen et al. (2010)]. However the consistency of the original LiNGAM method is lost in the extension.



- The simulated  $Y_t$  are obtained by applying :

$$Y_t^s = \sum_{h=0}^{\infty} \hat{\Phi}^h w_{t-h}^s,$$

with some truncation in practice.

- ii) The SVAR semi-parametric model

The simulations are more difficult when the nonfundamental representation concerns the autoregressive part. Indeed the process is mixed causal/noncausal, with a two sided moving average representation. However, the simulations are still rather easy to implement if we introduce the appropriate state variables summarizing the past and the future, respectively. This is done along the following steps [see Gourieroux, Jasiak (2014)];  
 First step : Compute the estimated roots of  $\det(Id - \hat{\Phi}z)$ . Some estimated roots are well-located (outside the unit circle), the others are ill-located (inside the unit circle).

Second step : Compute the adjoint matrix  $\tilde{\Phi}(L)$  of  $\hat{\Phi}(L) = Id - \hat{\Phi}L$ , that is the transpose of the matrix of cofactors. We have :

$$\tilde{\Phi}(L)\hat{\Phi}(L) = \det \hat{\Phi}(L)Id. \quad (4.17)$$

We can write :

$$\det \hat{\Phi}(L) = \hat{c}\hat{\varphi}(L)\hat{\psi}(L^{-1})L^{n-r}, \quad (4.18)$$

where  $\hat{\varphi}(L) = \prod_{j=1}^r (1 - \hat{\lambda}_j L)$ ,  $\hat{\lambda}_1, \dots, \hat{\lambda}_r$  being the well located roots,  $\hat{\psi}(L^{-1}) = \prod_{j=r+1}^n (1 - 1/\hat{\lambda}_j L^{-1})$ ,  $\hat{\lambda}_{r+1}, \dots, \hat{\lambda}_n$  being the ill-located roots, and  $\hat{c}$  the appropriate constant.

Third step : We have :

$$Y_t = \frac{1}{\hat{c}} \frac{\tilde{\Phi}(L)}{\hat{\varphi}(L)\hat{\psi}(L^{-1})} \varepsilon_{t+n-r} = \frac{1}{\hat{\varphi}(L)\hat{\psi}(L^{-1})} \xi_t, \quad (4.19)$$

with  $\xi_t = \frac{1}{\hat{c}} \tilde{\Phi}(L) \varepsilon_{t+n-r}$ .

The process  $(\xi_t)$  is a finite moving average, with possible ill-located roots, but is easy to simulate.

Let us introduce the processes  $(u_t)$  and  $(v_t)$  defined respectively by :

$$\hat{\psi}(L^{-1})u_t = \xi_t, \text{ and } \hat{\varphi}(L)v_t = \xi_t. \quad (4.20)$$

The process  $u_t$  (resp.  $v_t$ ) is function of the future components of  $\xi_t$  (resp. past components of  $\xi_t$ ), and we can write a partial fraction decomposition of

$$\frac{1}{\hat{\varphi}(L)\hat{\psi}(L^{-1})} = \frac{b_1(L)}{\hat{\varphi}(L)} + \frac{b_2(L)}{\hat{\psi}(L^{-1})}, \text{ say, where } b_1, b_2 \text{ are polynomials.}$$

Thus

$$Y_t = \left[ \frac{b_1(L)}{\hat{\varphi}(L)} + \frac{b_2(L)}{\hat{\psi}(L^{-1})} \right] \xi_t = b_1(L)v_t + b_2(L)u_t. \quad (4.21)$$

This means that the mixed process  $(Y_t)$  can be decomposed into lagged and leaded values of processes  $v_t$  and  $u_t$ , which are sufficient summaries of the past and future evolutions of  $\xi_t$ , respectively.

Fourth step : The simulated path is obtained as follows :

- Draw the  $\varepsilon_t^s$  independently in the sample distribution of the residuals  $\hat{\varepsilon}_t = Y_t - \hat{\Phi}Y_{t-1}$ .

- Deduce the simulated moving-average :  $\xi_t^s = \frac{1}{\hat{c}} \tilde{\Phi}(L)\varepsilon_{t+n-r}$ .

- Compute the simulated latent processes :

$$u_t^s = \frac{1}{\hat{\psi}(L^{-1})} \xi_t^s, \text{ by a forward expansion,}$$

$$v_t^s = \frac{1}{\hat{\varphi}(L)} \xi_t^s, \text{ by a backward expansion.}$$

- Finally the simulated path is :

$$Y_t^s = b_1(L)v_t^s + b_2(L)u_t^s.$$

The simulation technique above is based on the partial fraction decomposition and completely circumvents any Blaschke transformation.

## 5 Concluding remarks

We have shown in this paper that the identification difficulties in the analysis of SVARMA models are due to the poor performance of the estimation method used in the Box-Jenkins methodology, namely the Gaussian pseudo maximum likelihood. This approach suffers from the lack of identification existing in the Gaussian SVARMA. Whenever the shocks are not Gaussian, the SVARMA becomes identified up to change of scale, drift on time, and permutation of equations.

In fact a dynamic model constructed to derive impulse response functions requires much more structural assumptions on the error terms (i.e. independence) than a pure forecast model for which uncorrelated errors may be sufficient. In this respect the conventional econometric toolboxes available for macroeconomists have been conceived for a forecast purpose and are not appropriate for the analysis of policy shocks.

Moreover it is important to keep the structural VARMA dynamics and not to replace it by a VAR model with higher lags. Indeed simple consistent estimation methods can be introduced for nonfundamental representation in the moving average dynamics, and are rather different from the IV approaches used when the nonfundamental representation concerns the autoregressive dynamics. Similarly the methods to simulate a path of the process and to derive the impulse response function are very different if the autoregressive dynamics is causal or not. In particular nonlinear impulse response functions may have to be used when there is a nonfundamentality in the autoregressive dynamics.

Because of focusing on the second-order properties the SVARMA literature often introduces "incredible" identification assumptions that entail misspecification and naive interpretations of VARMA residuals. To paraphrase Sims (1980) : "Nonlinear analysis is getting easier, both because of improved techniques and because of better computational hardware. This weakens the excuse that second-order analysis has to be followed just since it is simple". Indeed misspecified analysis could not serve as a useful tool for economic policy. Nevertheless, as shown in the parametric and semi-parametric analysis developed in Section 4, SVARMA and SVAR can still be useful for economic policy, provided that the independence assumptions required for impulse response analysis are tested and the appropriate estimation methods are used.

The methods developed in this paper can be extended in several directions. First the asymptotic Gaussian distributions of the various estimators

proposed can be derived and testing procedures, in particular tests of fundamentalness, can be obtained. Second, the identification and estimation results might be extended to the case of more shocks than observables. Indeed identification results exist when the errors are not Gaussian [see e.g. Th 3.1. in Eriksson, Koivunen (2004) in the static case, or Gagliardini-Gourieroux (2014) for a non Gaussian factor model]. This possibility to identify the dynamics when  $m > n$  and the shocks are not Gaussian would be important in the discussion of the effect of omitted variables [see Giannone, Reichlin (2006), Lütkepohl (2014)].

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## Appendix 1

### What is an innovation ?

The term "innovation" is largely used in the applied literature, but often with different meanings and underlying assumptions. The aim of this appendix is to discuss this (these) notion (s). For expository purpose we consider a Markov process  $Y_t$ .

#### i) Definitions for a square integrable process.

Two different notions of innovations are considered in practice :

The weak linear innovation is :  $y_t = Y_t - EL(Y_t|Y_{t-1})$ , where  $EL(Y_t|Y_{t-1})$  is the best mean square approximation of  $Y_t$  by an affine function of  $Y_{t-1}$ .

The strong linear innovation is :  $u_t = Y_t - E(Y_t|Y_{t-1})$ , where  $E(Y_t|Y_{t-1})$  denotes the conditional expectation of  $Y_t$  given  $Y_{t-1}$ .

#### ii) Definition for an infinite variance process

In this case the notion of weak innovation makes no sense. But the notion of strong innovation still exists if the conditional distribution of  $Y_t$  given  $Y_{t-1}$  admits a first-order moments. This can occur when the process  $Y_t$  admits a first-order unconditional moment, but also sometimes for a process without a first-order unconditional moment, as shown with the example of the noncausal stable AR(1) process.

#### iii) Nonlinear innovation

The previous notions of innovation provide information on the accuracy of a prediction of  $Y_t$ , but no information when our interest is to predict a nonlinear transform of  $Y_t$ . A more appropriate notion is the nonlinear innovation  $\varepsilon_t$ , say, such that :

$$Y_t = g(Y_{t-1}, \varepsilon_t),$$

where the  $\varepsilon_t$  variables are i.i.d. standard normal.

This nonlinear autoregressive model is obtained as follows. Let us denote  $F(y|Y_{t-1})$  the conditional cdf of  $Y_t$  given  $Y_{t-1}$ . It is well-known that the variable :

$$U_t = F(Y_t|Y_{t-1})$$

follows a uniform distribution on  $[0, 1]$  for any  $Y_{t-1}$  and, in particular,  $U_t$  is independent of  $Y_{t-1}$ . By inverting the conditional cdf, we get :

$$Y_t = F^{-1}(U_t|Y_{t-1}),$$

and by defining  $\varepsilon_t = \Phi^{-1}(U_t)$ , where  $\Phi$  is the cdf of the standard normal we get :

$$Y_t = F^{-1}[\Phi(\varepsilon_t)|Y_{t-1}],$$

$$Y_t = g(Y_{t-1}, \varepsilon_t). \text{ (say),}$$

where  $g$  is strictly increasing in  $\varepsilon_t$ .

This notion of nonlinear innovation has three advantages :

- It can be used to evaluate the accuracy of the strong prediction of any nonlinear transform of  $Y_t$ .
- The  $\varepsilon_t$ 's are serially independent, whereas the weak and strong linear innovations are serially uncorrelated, but in general dependent. Thus, in finite sample, the weak and strong linear innovations cannot be used in general to model the shocks at the basis of impulse response analysis.
- By recursive substitutions, we deduce a nonlinear moving average representation of  $Y_t$  as function of independent innovations  $\varepsilon_t, \varepsilon_{t-1} \dots$  and by series expansion the Volterra representation of the Markov process (which generally differs from its Wold expansion).

## Appendix 2

### The main steps of the ML and Bayes procedures

**i) The ML procedure**

Let us detail this procedure for a VAR model :

$$Y_t - \Phi_1 Y_{t-1} \dots - \Phi_p Y_{t-p} = C \eta_t, \quad (\text{A.1})$$

say, where the  $\eta_t$ 's are i.i.d, with i.i.d. components. The common non-Gaussian distribution of the  $\eta_{j,t}$  can be parametrized. For instance, this might be a student distribution parametrized by the degree of freedom  $\nu$ .

For disentangling the causal and noncausal components of the stationary solution of (a.1), we have to consider the roots of  $\det \Phi(z)$ . For stationarity we assume that there are no roots on the unit circle.

$r$  roots are well-located and  $s = np - r$  ones are ill-located.

The expression of the likelihood function of model (A.1) depends on the number of ill-located roots and this number depends on the parameters of the model :  $s = s(\Phi)$ , say. Thus the log-likelihood function can be written as :

$$L(\Phi, C, \nu) = \sum_{s=0}^{np} \{L_s[\Phi, C, \nu] \mathbb{1}_{s(\Phi)=s}\}, \quad (\text{A.2})$$

where  $L_s(\Phi, C, \nu)$  is the expression of the log-likelihood function in regime  $s$ .

Since there are no roots on the unit circle, the subsets of parameters  $\Phi$  such that  $s(\Phi) = s$  are disjoint open sets. Moreover, due to the identifiability of the MA representation, there exists no distribution of process  $y$  belonging to two different regimes, and the standard ML theory applies.

The ML approach can be implemented along the following steps,

**step 1 :** Estimate the autoregressive order  $p$  by the standard BJ methodology, which provides a consistent estimation of the causal VAR representation of the process (which is a weak representation if the roots of  $\det \Phi(L)$  are not all outside the unit circle).

**step 2 :** Then maximize the log-likelihood function (A.2). Since the likelihood function is not differentiable due to the effects of the regime, this has to be done by first maximizing the regime specific log-likelihoods

$L_s$  under the constraints  $s(\Phi) = s$ , and then by selecting the regime  $s$  with the largest value of the regime specific log-likelihood function.

From a practical point of view, the difficulties come from :

i) the number of admissible regimes (but this number can be significantly diminished under structural constraints)

ii) the derivation of closed form expressions of the regime specific log-likelihoods [see e.g. Davis, Song (2012), or Lanne, Saikkonen (2013), in special cases].

## ii) The Bayes procedure

The standard Bayesian analysis of a VAR(1) model, say, consists in considering the likelihood function corresponding to the fundamental solution, that is,

$$l(\Phi, c, \nu) = \prod_{t=1}^T \frac{1}{|\det C|} g[C^{-1}(y_t - \Phi y_{t-1}), \nu],$$

where  $g(\cdot, \nu)$  denotes the joint pdf of independent variables with identical student distributions, say, and in completing by a prior on parameters  $\Phi, C, \nu$ . Usually this prior distribution attributes positive weights on value of  $\Phi$  such that some roots of  $\det(Id - \Phi z)$  are ill-located. This standard Bayesian approach will lead to inconsistent estimators for large  $T$ . Indeed, when some roots are ill-located, the Bayesian model assumes that  $(Y_t)$  may be non-stationary whereas the process is assumed strictly stationary in each regime in our framework.

Thus an appropriate Bayesian analysis has also to disentangle the different regimes as it is done in the ML approach. More precisely, the joint p.d.f. has to be written as :

$$l(\Phi, C, \nu) = \exp L(\Phi, C, \nu),$$

where  $L(\Phi, C, \nu)$  is given in (A.2), before specifying a prior, which can weight all the regimes.

## Appendix 3

### Identifiability, reversibility and estimation in a MA(1) process



For expository purpose, we consider the one-dimensional MA(1) process :  $y_t = \varepsilon_t - \theta\varepsilon_{t-1}$ , where the  $\varepsilon_t$ 's are independent. We first consider the asymptotic behaviour of the approximated maximum likelihood approach. Then we provide an example to illustrate the reason of identifiability in a non Gaussian case, and we consider a moment estimation method.

i) **Limit optimization problem in the approximate ML method**

We assume that the p.d.f. of the  $\varepsilon_t$ 's belongs to the family  $g(\varepsilon; \gamma)$ . The approximate log-likelihood function is :

$$\begin{aligned} L_T(\theta, \gamma) &= \mathbb{1}_{|\theta| < 1} \sum_{t=1}^T \log \left\{ g \left( \sum_{h=0}^{t-1} \theta^h y_{t-h}; \gamma \right) \right. \\ &\quad \left. + \mathbb{1}_{|\theta| > 1} \sum_{t=1}^T \log \left\{ \frac{1}{|\theta|} g \left( - \sum_{h=0}^{T-t-1} \frac{1}{\theta^{h+1}} y_{t+h+1}; \gamma \right) \right\} \right. \end{aligned}$$

When  $T$  goes to infinity  $\frac{1}{T}L_T$  converges to the limit function

$$\begin{aligned} L_\infty(\theta, \gamma) &= \mathbb{1}_{|\theta| < 1} E_0 \log g \left( \sum_{h=0}^{\infty} \theta^h y_{t-h}; \gamma \right) \\ &\quad + \mathbb{1}_{|\theta| > 1} E_0 \left[ \log \frac{1}{|\theta|} g \left( - \sum_{h=0}^{\infty} \frac{1}{\theta^{h+1}} y_{t+h+1}; \gamma \right) \right], \end{aligned}$$

where  $E_0$  is the expectation with respect to the true distribution of the process. We also have :

$$\begin{aligned} L_\infty(\theta, \gamma) &= \mathbb{1}_{|\theta| < 1} E_0 \log g[y_t - E_\theta(y_t | y_{-\infty}^{t-1}), \gamma] \\ &\quad + \mathbb{1}_{|\theta| > 1} E_0 \left\{ -\frac{1}{2} \log \theta^2 + \log g \left[ -\frac{1}{\theta} (y_{t+1} - E_\theta(y_{t+1} | y_{t+2}^\infty)); \gamma \right] \right\} \end{aligned}$$

with  $E_\theta(y_t | y_{-\infty}^{t-1}) = - \sum_{h=1}^{\infty} \theta^h y_{t-h}$  and  $E_\theta(y_{t+1} | y_{t+2}^\infty) = \sum_{h=1}^{\infty} \frac{1}{\theta^{h+1}} y_{t+h+1}$

In the Gaussian case, where the distribution of  $\varepsilon_t$  is  $N(0, \sigma^2)$ , we get :

$$\begin{aligned}
L_\infty(\theta, \sigma^2) &= \mathbb{1}_{|\theta| < 1} E_0 \left[ -\frac{1}{2} \log \sigma^2 - \frac{1}{2\sigma^2} (y_t - E_\theta(y_t | y_{-\infty}^{t-1}))^2 \right] \\
&+ \mathbb{1}_{|\theta| > 1} E_0 \left\{ -\frac{1}{2} \log(\theta^2 \sigma^2) - \frac{1}{2\sigma^2} \left[ \frac{1}{\theta^2} (y_{t+1} - E_\theta(y_{t+1} | y_{t+2}^\infty)) \right]^2 \right\}
\end{aligned}$$

The limit optimization problem is :

$$\min_{(\theta, \sigma^2)} [\mathbb{1}_{|\theta| < 1} L_1^a(\theta, \sigma^2) + \mathbb{1}_{|\theta| > 1} L_2^a(\theta, \sigma^2)]$$

$$\text{with } L_1^a(\theta, \sigma^2) = \log \sigma^2 + \frac{1}{\sigma^2} E_0 (y_t - E_\theta(y_t | y_{-\infty}^{t-1}))^2,$$

$$L_2^a(\theta, \sigma^2) = \log(\theta^2 \sigma^2) + \frac{1}{\theta^2 \sigma^2} E_0 (y_{t+1} - E_\theta(y_{t+1} | y_{t+2}^\infty))^2.$$

Due to the reversibility of the Gaussian process the true distribution of  $y_{t+1} - E_\theta(y_{t+1} | y_{t+2}^\infty)$  if  $|\theta| > 1$  is the same as the true distribution of  $y_{t+1} - E_{1/\theta}(y_{t+1} | y_{-\infty}^t)$ .

Let us assume that  $|\theta_0| < 1$  and let us consider the solutions of the limit optimization problem.

In order to minimize  $L_1(\theta, \sigma^2)$  on  $|\theta| < 1$  we can concentrate with respect to  $\sigma^2$  and we get :

$$\min_{\theta} \log E_0 [y_t - E_\theta(y_t | y_{-\infty}^{t-1})]^2 + 1.$$

The minimum is reached for  $\theta = \theta_0$  and the value at the minimum is :  $\log \sigma_0^2 + 1$ .

In order to minimize  $L_2(\theta, \sigma^2)$  on  $|\theta| > 1$ , we can put  $\theta^2 \sigma^2 = \tilde{\sigma}^2$ ; we get

$$\min_{\theta} \log E_0 [y_{t+1} - E_{1/\theta}(y_{t+1} | y_{-\infty}^t)] + 1.$$

The minimum is reached for  $\theta = \frac{1}{\theta_0}$  and the minimum is again  $\log \sigma_0^2 + 1$ .

When  $|\theta_0|$  is larger than 1, we can see that  $L_1(\theta, \sigma^2)$  is optimal for  $\frac{1}{\theta_0}$  and  $L_2(\theta, \sigma^2)$  for  $\theta_0$  and we still have two inverse values of  $\theta$  giving the same optimum namely  $\log(\theta_0^2 \sigma_0^2) + 1$ . The model is not asymptotically identifiable.

Note however that, in finite sample, the optimal values of  $L_1^a$  and  $L_2^a$  are different, even in the Gaussian case. Thus the approximated ML approach will provide a unique solution, not necessarily well-located.

## ii) Identification in the non-Gaussian case

Let us consider the joint distribution of  $(y_t, y_{t-1})$ . The characteristic function of this distribution is :

$$\begin{aligned}\psi(u, v) &= E \exp[i(uy_t + vy_{t-1})] \\ &= E \exp(iu\varepsilon_t) E \exp[i(v - u\theta)\varepsilon_{t-1}] E[\exp(-iv\theta\varepsilon_{t-2})].\end{aligned}$$

Let us for instance assume that  $\varepsilon_t$  follows a stable distribution, we get :

$$\psi(u, v) = \exp[-c(|u|^\alpha + |v - u\theta|^\alpha + |v\theta|^\alpha)]$$

Is this function of  $(c, \theta)$  injective ?

If  $\alpha = 2$ , i.e. in the Gaussian case, we verify that :

$$c[u^2 + (v - u\theta)^2 + v^2\theta^2] = c[(u^2 + v^2)(1 + \theta^2) - 2uv\theta]$$

takes the same value for  $(c, \theta)$  and  $(c\theta^2, \frac{1}{\theta})$  and we do not have identifiability. On the contrary for  $\alpha \neq 2$ , we see, for instance, that  $\psi(u, v)$  is not differentiable in  $u = 0, v = 0$  and  $\theta = v/u$ , and the latter condition implies the identifiability of  $\theta$ .

## iii) Moment method

If we do not want to make a parametric assumption about the distribution of  $\varepsilon_t$  we can use a moment method based on higher order cross moments.

Let us consider again the one-dimensional MA(1) process. We have :

$$E(y_t y_{t-1}^2) = -\theta E\varepsilon_t^3,$$

$$E(y_t^2 y_{t-1}) = \theta^2 E\varepsilon_t^3,$$

and therefore :

$$\theta = -\frac{E(y_t^2 y_{t-1})}{E(y_t y_{t-1}^2)},$$

whenever  $\varepsilon_t$  has a skewed distribution, i.e.  $E(\varepsilon_t^3) \neq 0$ . Thus the location of  $\theta$  w.r.t. 1 is identified from the no time reversibility of the process.